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Integrable twists in AdS/CFT

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Abstract: A class of marginal deformations of four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory has been found to correspond to a set of smooth, multiparameter deformations of the S^5 target subspace in the holographic dual on $AdS_5 \times S^5$. We present here an analogous set of deformations that act on global toroidal isometries in the AdS_5 subspace. Remarkably, certain sectors of the string theory remain classically integrable in this larger class of so-called γ-deformed $AdS_5 \times S^5$ backgrounds. Relying on studies of deformed $\mathfrak{su}(2)_\gamma$ models, we formulate a local $\mathfrak{sl}(2)_\gamma$ Lax representation that admits a classical, thermodynamic Bethe equation (based on the Riemann-Hilbert interpretation of Bethe's ansatz) encoding the spectrum in the deformed AdS_5 geometry. This result is extended to a set of discretized, asymptotic Bethe equations for the twisted string theory. Near-pp-wave energy spectra within $\mathfrak{sl}(2)_\gamma$ and $\mathfrak{su}(2)_\gamma$ sectors provide a useful and stringent test of such equations, demonstrating the reliability of this technology in a wider class of string backgrounds. In addition, we study a twisted Hubbard model that yields certain predictions of the dual β-deformed gauge theory.

Keywords: Gauge-gravity correspondence, Integrable Field Theories, AdS-CFT Correspondence.

Contents

1. Introduction

In [1], Lunin and Maldacena used an SL$(3,R)$ deformation of $AdS_5 \times S^5$ to find a supergravity solution dual to a class of marginal deformations (known as Leigh-Strassler [2] or β-deformations) of $\mathcal{N}=4$ super Yang-Mills (SYM) theory. This provided an interesting opportunity to study the AdS/CFT correspondence [3−5] in new gravity backgrounds with less supersymmetry. In the case of real deformations, one obtains the gravity dual of a one-parameter family of $\mathcal{N}=1$ conformal gauge theories, and this particular example has been the focus of many recent investigations: pp-wave limits were studied in [6, 7], for example, and other interesting string systems were examined in [8−14].

The notion of the Lunin-Maldacena deformation was generalized by Frolov [15] by considering a sequence of T-dualities and coordinate shifts, or TsT deformations, acting on global toroidal isometries in S^5. By parameterizing each TsT deformation with separate $\tilde{\gamma}_i$ ($i \in 1, 2, 3$), one generically obtains a non-supersymmetric theory, dual to a non-supersymmetric deformation of $\mathcal{N} = 4$ SYM.[1] (Adhering to conventions in the literature, we will use the symbols $\tilde{\gamma}_i$ to indicate deformation parameters that naturally appear in the background geometry.) This construction can be extended to include complex deformations by including SL$(2, R)$ transformations. By studying string theory on AdS_5 backgrounds with TsT-deformed S^5 factors, Frolov was also able to demonstrate that bosonic string solutions in these backgrounds can be generated by imposing twisted boundary conditions on known solutions in the undeformed $AdS_5 \times S^5$ geometry. The full action for Green-Schwarz strings in TsT-deformed backgrounds was subsequently constructed in [16], where it was shown that *superstring* solutions in such backgrounds are again mapped (in a one-to-one fashion) from solutions in the parent geometry, deformed by twisted boundary conditions.

Another interesting property of TsT transformations in S^5 is that the integrability of classical string theory on $AdS_5 \times S^5$ [17, 18] seems to be preserved under these deformations. In [9], Frolov, Roiban and Tseytlin were able to derive classical Bethe equations encoding the spectral problem in (classically) closed sectors on the deformed S^5 subspace. Similar to the undeformed case [19 – 24], Frolov [15] was subsequently able to derive a Lax representation for the bosonic sector of the deformed-S^5 theory: the essential observation was that one can gauge away the non-derivative dependence of the Lax representation on the U(1) isometry fields involved in the deformation.

In addition to deriving twisted Bethe equations, Frolov, Roiban and Tseytlin demonstrated in [9] that more general fast-string limits in these deformed backgrounds can be described by a Landau-Lifshitz action corresponding to a continuum limit of anisotropic spin chains associated with the scalar sector of the deformed $\mathcal{N} = 1$ dual gauge theory[2] [25, 26]. Various aspects of these twisted spins chains have been studied in [11, 27 – 29], for example. In this vein, Beisert and Roiban provided a detailed study of related spin-chain systems with a variety of twists in [30] that will be particularly useful in the present context.

In this paper we add to previous studies of semiclassical strings in γ-deformed backgrounds. We focus largely on deformations of the AdS_5 subspace analogous to Frolov's multiparameter TsT deformations. We find a one-parameter family of such deformations that can be understood as a usual TsT deformation acting on a global U(1) $\times$ U(1) isometry in AdS_5, while a wider class of deformations can be seen as arising from TsT transformations that involve T-duality along timelike directions. As with the S^5 deformations, however, we can again interpret such transformations as formally giving rise to twisted boundary conditions from the perspective of the undeformed theory, and integrability in the classical string theory is again preserved. In these new geometries we find a number of peculiar features, and we expect the dual gauge theory to be modified dramatically (perhaps to a non-commutative gauge theory, along the lines of [31, 32]).

In section 2 we establish notation, review TsT deformations on the S^5 subspace and parameterize the geometry in a way that is convenient for studying a pp-wave limit of the

[1] Of course, the $\tilde{\gamma}_i$ can be chosen to reproduce the Lunin-Maldacena solution as a special case.

[2] Notions of integrability on the gauge theory side of the correspondence are understood here to be restricted to the planar limit.

deformed background. In section 3 we present analogous deformations of the AdS_5 subspace and study properties of the resulting geometry. We study a suitable Lax representation for string theory on this background in section 4, and compute the Riemann-Hilbert formulation of the classical Bethe equations within deformed $\mathfrak{sl}(2)_\gamma$ sectors of the theory. Energy spectra in the near-pp-wave limit of γ-deformed $\mathfrak{su}(2)_\gamma$ and $\mathfrak{sl}(2)_\gamma$ sectors are computed in section 5. In section 6 we study γ-deformed, discrete extrapolations of the thermodynamic string Bethe equations, and comment on the ability of these equations to reproduce the near-pp-wave energy spectra of BMN strings in these backgrounds. In section 7 we study a twisted Hubbard model, analogous to [33], that is conjectured to yield the deformed $\mathfrak{su}(2)_\gamma$ sector of the dual gauge theory. In the case of the corresponding deformed $\mathfrak{sl}(2)_\gamma$ sector, we have little to say regarding predictions from the gauge theory side of the correspondence. In the final section 8, however, we comment on various relevant Bethe equations proposed in [30].

2. Geometry and TsT deformations

TsT transformations correspond to a sequence of worldsheet duality transformations, and one expects deformed backgrounds obtained in this manner to be exact solutions of the equations of motion. By including S-duality one can extend this class of transformations (parameterized by real $\tilde{\gamma}_i$) to include complex deformation parameters. (Backgrounds obtained in this fashion, however, are expected to be modified by α'/R^2 corrections [9].) For the case of real $\tilde{\gamma}_i$, the deformed spacetime metric and relevant background fields are given by (mostly following the notation of [34]),

$$\begin{aligned} ds^2_{\text{string}}/R^2 &= ds^2_{AdS_5} + \sum_{i=1}^{3}(d\rho_i^2 + G\rho_i^2 d\phi_i^2) + G\rho_1^2\rho_2^2\rho_3^2[d(\sum_{i=1}^{3}\tilde{\gamma}_i\phi_i)]^2\,, \\ B_2 &= R^2 G w_2\,, \qquad e^{\phi} = e^{\phi_0}G^{1/2}\,, \qquad \chi = 0\,, \\ w_2 &\equiv \tilde{\gamma}_3\,\rho_1^2\rho_2^2\,d\phi_1\wedge d\phi_2 + \tilde{\gamma}_1\,\rho_2^2\rho_3^2\,d\phi_2\wedge d\phi_3 + \tilde{\gamma}_2\,\rho_3^2\rho_1^2\,d\phi_3\wedge d\phi_1\,, \\ G^{-1} &\equiv 1 + \tilde{\gamma}_3^2\,\rho_1^2\rho_2^2 + \tilde{\gamma}_1^2\,\rho_2^2\rho_3^2 + \tilde{\gamma}_2^2\,\rho_1^2\rho_3^2\,, \end{aligned} \tag{2.1}$$

where we have set $\alpha' = 1$ for convenience. The component $ds^2_{AdS_5}$ represents the undeformed metric on the AdS_5 subspace, while the S^5 subspace has undergone three consecutive TsT deformations parameterized by the $\tilde{\gamma}_i$. To be certain, the $\tilde{\gamma}_i$ appearing in the metric parameterize the coordinate-shift part of individual TsT deformations ($\phi_1 \to \phi_1 + \tilde{\gamma}_i\,\phi_2$, for example). B_2 is the NS-NS two-form field strength (we have omitted the two- and five-form field strengths C_2 and F_5). The usual angle variables on the sphere can be encoded in the convenient parameterization

$$\rho_1 = \sin\alpha\cos\theta\,, \quad \rho_2 = \sin\alpha\sin\theta\,, \quad \rho_3 = \cos\alpha\,, \tag{2.2}$$

such that $\sum_{i=1}^{3}\rho_i^2 = 1$. The string coupling g_s is related to the gauge theory coupling g_{YM} via the standard relation, $g_s = e^{\phi_0} = g^2_{\text{YM}}/4\pi$, and the radial scale of both the AdS_5 and S^5 spaces is given by $R^4 = 4\pi g_s N_c = g^2_{\text{YM}}N_c \equiv \lambda$, where N_c is the rank of the Yang-Mills gauge group. We will restrict our attention to the full planar limit $N_c \to \infty$.

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For present purposes, we find it convenient to introduce the following alternative parameterization on S^5:

$$\rho_2 = \frac{y_1}{R}\,, \qquad \rho_3 = \frac{y_2}{R}\,, \qquad \rho_1 = \sqrt{1-\rho_2^2-\rho_3^2}\,, \qquad \phi_1 = x^+ + \frac{x^-}{R^2}\,, \qquad t = x^+\,. \tag{2.3}$$

This choice of lightcone coordinates implies that, as R becomes large, we approach a semiclassical limit described by point-like (or "BMN" [35]) strings boosted to lightlike momentum J along a geodesic on the deformed S^5. The angular momentum J in the ϕ_1 direction is related to the scale radius R according to

$$p_- R^2 = J\,, \tag{2.4}$$

and the lightcone momenta take the form

$$-p_+ = \Delta - J\,, \qquad -p_- = i\partial_{x^-} = \frac{i}{R^2}\partial_\phi = -\frac{J}{R^2}\,. \tag{2.5}$$

At this stage we find it convenient to use the following form of the AdS_5 metric:

$$ds^2_{AdS_5} = -\left(\frac{1+x^2/4R^2}{1-x^2/4R^2}\right)^2 dt^2 + \frac{dx^2/R^2}{(1-x^2/4R^2)^2}\,. \tag{2.6}$$

This version of the spacetime metric is also useful when working with fermions (see [36, 37] for details), though we will restrict ourselves to the bosonic sector of the string theory in the present study. The coordinates x^k, $y_1^{k'}$ and $y_2^{k'}$ span an $\mathrm{SO}(4)\times\mathrm{SO}(2)\times\mathrm{SO}(2)$ transverse space, with z_k lying in AdS_5 ($k \in 1,\ldots,4$), and $y_1^{k'}$, $y_2^{k'}$ parameterizing the deformed S^5 subspace ($k' \in 1,2$). In the pp-wave limit, p_- is held fixed while J and R become infinite, and the planar limit is taken such that the quantity N_c/J^2 is held fixed. The lightcone momentum p_- is then equated with

$$p_- = \frac{1}{\sqrt{\lambda'}} = \frac{J}{\sqrt{g_{\mathrm{YM}}^2 N_c}}\,, \tag{2.7}$$

where λ' is known as the modified 't Hooft coupling, and J is equated on the gauge theory side with a scalar component of the SU(4) R-charge.

For reasons described in [36, 37], the lightcone coordinates in eq. (2.3) admit many simplifications, the most important of which is the elimination of normal-ordering contributions to the lightcone Hamiltonian. (More recently, however, a "uniform" lightcone gauge choice has proved to be useful in certain contexts [38].) It is convenient to introduce the following complex coordinates

$$\begin{aligned} y &= y_1\cos\phi_2 + iy_1\sin\phi_1\,, \qquad & \bar{y} &= y_1\cos\phi_2 - iy_1\sin\phi_1\,, \\ z &= y_2\cos\phi_2 + iy_2\sin\phi_1\,, \qquad & \bar{z} &= y_2\cos\phi_2 - iy_2\sin\phi_1\,. \end{aligned} \tag{2.8}$$

Arranging the large-R expansion of the spacetime metric according to

$$ds^2 = ds^2_{(0)} + \frac{ds^2_{(1)}}{R^2} + O(1/R^4)\,, \tag{2.9}$$

we therefore find

$$\begin{aligned}
ds^2_{(0)} &= 2dx^+dx^- + |dy|^2 + |dz|^2 - (dx^+)^2\left[x^2 + |y|^2(1+\tilde\gamma_3^2) + |z|^2(1+\tilde\gamma_2^2)\right] , \\
ds^2_{(1)} &= (dx^-)^2 + \frac{1}{4}(yd\bar y + \bar y dy + zd\bar z + \bar z dz)^2 - 2dx^+dx^-(|y|^2(1+\tilde\gamma_3^2) + |z|^2(1+\tilde\gamma_2^2)) \\
&\quad + \frac{1}{2}x^2dx^2 + (dx^+)^2\Big[(-\frac{1}{2}x^4 + 2(|z|^2+|y|^2)(|z|^2\tilde\gamma_2^2 + |y|^2\tilde\gamma_3^2) + (|y|^2\tilde\gamma_3^2 + |z|^2\tilde\gamma_2^2)^2\Big] \\
&\quad + \tilde\gamma_1 dx^+(\tilde\gamma_2|z|^2\Im(\bar y dy) + \tilde\gamma_3|y|^2\Im(\bar z dz)) - \tilde\gamma_3^2\Im(\bar y dy)^2 - \tilde\gamma_3^2\Im(\bar z dz)^2 \\
&\quad + \tilde\gamma_2\tilde\gamma_3\Im(\bar y dy)\Im(\bar z dz) .
\end{aligned} \tag{2.10}$$

At leading order one obtains a pp-wave metric, with obvious contributions from the deformed and undeformed subspaces. At both leading and sub-leading order near the pp-wave limit, we find that the geometry is deformed only by the parameters $\tilde\gamma_2$ and $\tilde\gamma_3$ (this is just a consequence of the particular semiclassical limit we have chosen). The corresponding expansion of the NS-NS two-form B_2 appears as

$$\begin{aligned}
B_2 &= \tilde\gamma_3 dx^+\wedge\Im(\bar y dy) - \tilde\gamma_2 dx^+\wedge\Im(\bar z dz) \\
&\quad + \frac{1}{R^2}\left[-\tilde\gamma_3(\tilde\gamma_3^2|y|^2 + \tilde\gamma_2^2|z|^2)dx^+\wedge\Im(\bar y dy) + \tilde\gamma_2(\tilde\gamma_3^2|y|^2 + \tilde\gamma_2^2|z|^2)dx^+\wedge\Im(\bar z dz)\right. \\
&\quad \left. + \tilde\gamma_3 dx^-\Im(\bar y dy) - \tilde\gamma_2 dx^-\wedge\Im(\bar z dz) + \tilde\gamma_1\Im(\bar y dy)\wedge\Im(\bar z dz)\right] .
\end{aligned} \tag{2.11}$$

We truncate to $\mathfrak{su}(2)_\gamma$ sectors of the geometry by projecting onto a single complex coordinate, which isolates a one-parameter TsT deformation:

$$\begin{aligned}
ds^2_{\mathfrak{su}(2)_\gamma} &= 2dx^+dx^- - \left(1+\tilde\gamma^2\right)|y|^2(dx^+)^2 + |dy|^2 + \frac{1}{R^2}\Big[\frac{1}{4}(yd\bar y + \bar y dy)^2 \\
&\quad + (dx^-)^2 + \tilde\gamma^2(2+\tilde\gamma^2)|y|^4(dx^+)^2 - 2(1+\tilde\gamma^2)|y|^2dx^+dx^- - \tilde\gamma\Im\left(\bar y dy\right)\Big] + O(1/R^4) .
\end{aligned} \tag{2.12}$$

The parameter $\tilde\gamma$ here can stand for either $\tilde\gamma_2$ or $\tilde\gamma_3$, corresponding to two possible choices of $\mathfrak{su}(2)_\gamma$ truncation. The NS-NS two-form reduces in this $\mathfrak{su}(2)_\gamma$ sector to

$$\begin{aligned}
B_2^{\mathfrak{su}(2)_\gamma} &= \tilde\gamma dx^+\wedge\Im(\bar y dy) + \frac{\tilde\gamma}{R^2}\Big(dx^-\wedge\Im(\bar y dy) \\
&\quad - (1+\tilde\gamma^2)|y|^2dx^+\wedge\Im(\bar y dy)\Big) + O(1/R^4) .
\end{aligned} \tag{2.13}$$

Given this parameterization we can easily calculate the string lightcone Hamiltonian and solve for its (semiclassical) spectrum. Prior to attacking this problem, we will formulate an analogous deformation on the AdS_5 subspace.

3. Deformations of AdS_5

To study TsT deformations on AdS_5, we find it convenient to parameterize the geometry in a nearly identical fashion to the S^5 case above, such that the spacetime metric in each individual subspace is mapped into the other under an obvious Wick rotation. It is useful in this regard to start with an SO(4, 2) invariant, expressed in terms of $\mathbb{R}^6$ embedding coordinates,

$$- X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 - X_5^2 = -1\,, \tag{3.1}$$

which we write as

$$\begin{aligned} X_0 &= \eta_1 \sin\hat\varphi_1\,, & X_1 &= \eta_2 \cos\hat\varphi_2\,, \\ X_2 &= \eta_2 \sin\hat\varphi_2\,, & X_3 &= \eta_3 \cos\hat\varphi_3\,, \\ X_4 &= \eta_3 \sin\hat\varphi_3\,, & X_5 &= \eta_1 \cos\hat\varphi_1\,. \end{aligned} \tag{3.2}$$

The hatted notation $\hat\varphi_i$ is employed to denote untwisted U(1) angular coordinates. This formulation is connected with the usual angular variables on AdS_5 under the assignment

$$\eta_1 = \cosh\alpha\,, \qquad \eta_2 = \sinh\alpha\sin\theta\,, \qquad \eta_3 = \sinh\alpha\cos\theta\,, \tag{3.3}$$

which preserves the SO(2,1) invariant

$$- \eta_1^2 + \eta_2^2 + \eta_3^2 = -1\,. \tag{3.4}$$

One thereby obtains the more familiar spacetime metric:

$$\begin{aligned} ds^2_{AdS_5}/R^2 &= -(d\eta_1^2 + \eta_1^2 d\hat\varphi_1^2) + \sum_{i=2}^{3}(d\eta_i^2 + \eta_i^2 d\hat\varphi_i^2) \\ &= d\alpha^2 - \cosh\alpha^2 d\hat\varphi_1^2 + \sinh\alpha^2\left(d\theta^2 + \sin\theta^2 d\hat\varphi_2^2 + \cos\theta^2 d\hat\varphi_3^2\right)\,. \end{aligned} \tag{3.5}$$

This metric exhibits a manifest U(1) × U(1) × U(1) global symmetry: the deformations of interest thus act on the corresponding angular coordinates $\hat\varphi_i$ ($i \in 1,2,3$). At this stage one may be concerned that invoking T-duality on compact timelike directions will lead to complications. While this concern will be addressed below, we aim to simplify the discussion by focusing on a single TsT deformation for which this issue can be avoided under certain assumptions. Following any manipulations, of course, one must pass to the universal covering space in which the global time coordinate in the resulting geometry is understood to be noncompact.

3.1 Single-parameter TsT deformation

It turns out that TsT transformations on the angular pair $(\hat\varphi_2, \hat\varphi_3)$ result in a deformation that is trivial upon reduction to deformed $\mathfrak{sl}(2)_\gamma$ subsectors of the theory. The global U(1)× U(1) isometry of interest is thus chosen to be parameterized by the angular coordinates $\hat\varphi_1$ and $\hat\varphi_2$. The $\hat\varphi_2$ direction is spacelike, so we invoke a TsT transformation that acts as a T-duality along the $\hat\varphi_2$ direction, a shift in the $\hat\varphi_1$ direction $\hat\varphi_1 \to \hat\varphi_1 + \tilde\gamma\hat\varphi_2$, followed by a second T-duality in the new φ_2 direction (where φ_2 now stands for a "transformed" angular coordinate). The deformed spacetime metric thus takes the form

$$ds^2_{\rm str}/R^2 = ds^2_{S^5} + g^{ij}d\eta_i d\eta_j + g^{ij}\,G\,\eta_i^2 d\varphi_j^2 - \tilde\gamma^2\,G\,\eta_1^2\eta_2^2\eta_3^2\,d\varphi_1^2\,, \tag{3.6}$$

where $g = \mathrm{diag}(-1, 1, 1)$, and the deformation factor G is given by

$$G^{-1} \equiv 1 - \tilde{\gamma}^2 \eta_1^2 \eta_2^2 \,. \tag{3.7}$$

The NS-NS two-form appears as

$$B_2/R^2 = \tilde{\gamma}\, G\, \eta_1^2 \eta_2^2 \, d\varphi_1 \wedge d\varphi_2 \,. \tag{3.8}$$

An obvious concern is that one has generated spacetime directions in this background that exhibit mixed signature. In fact, immediately following the shift in the $\hat{\phi}_1$ direction, the dualized ϕ_2 direction becomes mixed. From the worldsheet perspective, the string action should only be sensitive to the local region of the target space in which the string propagates, so we may choose to study the theory in the region where ϕ_2 is strictly spacelike. In any case, we find it efficient (for the moment) to proceed pragmatically by studying the theory in regions of the geometry where the signature is unambiguous. It will be shown below that this naive approach yields a lightcone Hamiltonian that appears to be sensible for our purposes.

Under the deformation in eq. (3.6), the $AdS_5^{\tilde{\gamma}}$ worldsheet action can be written as

$$\begin{aligned} S_{AdS_5^{\tilde{\gamma}}} = -\frac{\sqrt{\lambda}}{2} \int d\tau \frac{d\sigma}{2\pi} \Big[& \gamma^{\alpha\beta} \left(g^{ij} \partial_\alpha \eta_i \partial_\beta \eta_j + g^{ij}\, G\, \eta_i^2 \partial_\alpha \varphi_j \partial_\beta \varphi_j - \tilde{\gamma}^2\, G\, \eta_1^2 \eta_2^2 \eta_3^2\, \partial_\alpha \varphi_1 \partial_\beta \varphi_1 \right) \\ & - 2\epsilon^{\alpha\beta} \left(\tilde{\gamma}\, G\, \eta_1^2 \eta_2^2\, \partial_\alpha \varphi_1 \partial_\beta \varphi_2 + \Lambda (g^{ij} \eta_i \eta_j + 1) \right) \Big] \,, \end{aligned} \tag{3.9}$$

where Λ acts as a Lagrange multiplier enforcing eq. (3.4) on shell. The indices α and β run over the τ ($\alpha,\ \beta = 0$) and σ ($\alpha,\ \beta = 1$) directions on the worldsheet, and $\gamma^{\alpha\beta}$ is the worldsheet metric. With the intent of studying semiclassical limits of this action, it is convenient to choose a lightcone coordinate parameterization analogous to eq. (2.3) above:

$$\eta_2 = \frac{u_1}{R} \,, \qquad \eta_3 = \frac{u_2}{R} \,, \qquad \eta_1 = \sqrt{1_{\eta_2}{}^2 + \eta_3^2} \,, \qquad \phi_1 = x^+ + \frac{x^-}{R^2} \,, \quad t = x^+ \,, \tag{3.10}$$

and rewrite the metric on S^5 (with an SO(4) coordinate s) as

$$ds^2_{S^5} = \left(\frac{1 - s^2/4R^2}{1 + s^2/4R^2} \right)^2 dt^2 + \frac{ds^2/R^2}{(1 - s^2/4R^2)^2} \,. \tag{3.11}$$

By analogy with eq. (2.8), we introduce the following complex coordinates

$$\begin{aligned} v &= u_1 \cos\varphi_2 + i u_1 \sin\varphi_1 \,, \qquad & \bar{v} &= u_1 \cos\varphi_2 + i u_1 \sin\varphi_1 \,, \\ w &= u_2 \cos\varphi_2 + i u_2 \sin\varphi_1 \,, \qquad & \bar{w} &= u_2 \cos\varphi_2 + i u_2 \sin\varphi_1 \,, \end{aligned} \tag{3.12}$$

so that truncation to deformed $\mathfrak{sl}(2)_\gamma$ sectors involves projecting onto the coordinate pairs $(v, \bar{v})$ or $(w, \bar{w})$. In the $(v, \bar{v})$ projection the large-R expansion of the spacetime metric and NS-NS two-form yields:

$$\begin{aligned} ds^2_{\mathfrak{sl}(2)_\gamma} &= 2dx^+ dx^- - (1 + \tilde{\gamma}^2)|v|^2 (dx^+)^2 + |dv|^2 - \frac{1}{R^2} \Big[\frac{1}{4} (v d\bar{v} + \bar{v} dv)^2 - (dx^-)^2 \\ &\quad + \tilde{\gamma}^2 (2 + \tilde{\gamma}^2) |v|^4 (dx^+)^2 + \tilde{\gamma}^2 (v d\bar{v} - \bar{v} dv)^2 \Big] + O(1/R^4) \,, \\ B_2^{\mathfrak{sl}(2)_\gamma} &= \frac{i}{2} \tilde{\gamma} dx^+ \wedge (v d\bar{v} - \bar{v} dv) + \frac{i}{2R^2} |v|^2 \tilde{\gamma} (1 + \tilde{\gamma}^2) dx^+ \wedge (\bar{v} dv - v d\bar{v}) + O(1/R^4) \,. \end{aligned} \tag{3.13}$$

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As with the corresponding deformations on S^5, the TsT deformation considered here amounts to a set of twisted boundary conditions on the undeformed U(1) coordinates. As in [15], one finds that the conserved U(1) currents J_i^α in the undeformed theory are identical to those in the deformed theory. Labeling canonical momenta as $p_i = J_i^0$, the associated charges take the form

$$J_i = \int \frac{d\sigma}{2\pi} p_i \,. \tag{3.14}$$

One therefore finds that the identification of the deformed and undeformed currents J_i^1 leads to the conditions

$$\begin{aligned} \hat{\varphi}_1' &= \varphi_1' - \gamma\, p_2 \,, \\ \hat{\varphi}_2' &= \varphi_2' + \gamma\, p_1 \,, \\ \hat{\varphi}_3' &= \varphi_3' \,, \end{aligned} \tag{3.15}$$

where φ' denotes a worldsheet σ derivative acting on φ, and, for convenience (and to remain consistent with the literature), we have introduced the rescaled deformation parameter

$$\gamma \equiv \frac{\tilde{\gamma}}{\sqrt{\lambda}} \,. \tag{3.16}$$

In the case of S^5 deformations, this quantity is to be identified with β, which is the deformation parameter in the corresponding β-deformed gauge theory.

These equations imply the following twisted boundary conditions on the undeformed U(1) coordinates:

$$\begin{aligned} \hat{\varphi}_1(2\pi) - \hat{\varphi}_1(0) &= 2\pi(m_1 - \gamma\, J_2) \,, \\ \hat{\varphi}_2(2\pi) - \hat{\varphi}_2(0) &= 2\pi(m_2 + \gamma\, J_1) \,, \\ \hat{\varphi}_3(2\pi) - \hat{\varphi}_3(0) &= 2\pi m_3 \,, \end{aligned} \tag{3.17}$$

where m_i here denotes integer winding numbers defined by

$$2\pi m_i = \varphi_i(2\pi) - \varphi_i(0) \,. \tag{3.18}$$

In other words, the TsT deformation in eq. (3.6) amounts to imposing the above boundary conditions on the angular variables $\hat{\varphi}_i$, in precise analogy with corresponding deformations of S^5.

3.2 Euclideanized deformation

It was suggested in [30] that the above deformation of AdS_5 might correspond to a non-commutative deformation of the dual Yang-Mills theory. To explore this possibility, it is useful to consider TsT deformations in Euclidean AdS_5. Starting from a Wick rotation of eq. (3.1)

$$-X_0^2 + X_5^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 = -1 \,, \tag{3.19}$$

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with the parameterization

$$\begin{aligned} X_0 &= \eta_1 \cosh \hat{\varphi}_1\,, \qquad & X_5 &= \eta_1 \sinh \hat{\varphi}_1\,, \\ X_1 &= \eta_2 \cos \hat{\varphi}_2\,, \qquad & X_2 &= \eta_2 \sin \hat{\varphi}_2\,, \\ X_3 &= \eta_3 \cos \hat{\varphi}_3\,, \qquad & X_4 &= \eta_3 \sin \hat{\varphi}_3\,, \end{aligned} \tag{3.20}$$

one obtains the following undeformed Euclideanized spacetime metric:

$$\frac{ds^2}{R^2} = -d\eta_1^2 + \eta_1^2 d\hat{\varphi}_1^2 + \sum_{i=2}^{3} (d\eta_i^2 + \eta_i^2 d\hat{\varphi}_i^2)\,. \tag{3.21}$$

A TsT transformation in the $(\hat{\varphi}_1, \hat{\varphi}_2)$ coordinates yields

$$ds^2/R^2 \;=\; -d\eta_1^2 + G\eta_1^2 d\varphi_1^2 + \sum_{i=2}^{3} (d\eta_i^2 + G\eta_i^2 d\varphi_i^2) - G\;\eta_1^2\eta_2^2\eta_3^2\;\tilde{\gamma}^2 d\varphi_3^2\,, \tag{3.22}$$

with $G^{-1} = 1 + \tilde{\gamma}^2 \eta_1^2 \eta_2^2$, and $B_{12} = \tilde{\gamma}\;G\;\eta_1^2\eta_2^2$. Moving to the Poincaré coordinates

$$\eta_1 = \frac{\sqrt{y^2 + z_1^2 + z_2^2}}{y}\,, \qquad \eta_2 = \frac{z_1}{y}\,, \qquad \eta_3 = \frac{z_2}{y}\,, \qquad \varphi_1 = \ln\sqrt{y^2 + z_1^2 + z_2^2}\,, \tag{3.23}$$

we obtain the following metric:

$$\begin{aligned} ds^2/R^2 \;=\; & \frac{1}{y^2}\left(dy^2 + dz_1^2 + dz_2^2\right) + \frac{G}{y^2}\left(z_1^2 d\varphi_2^2 + z_2^2 d\varphi_3^2\right) - \frac{G\tilde{\gamma}^2}{y^6}(y^2 + z_1^2 + z_2^2) z_1^2 z_2^2 d\varphi_3^2 \\ & - \left(\frac{\tilde{\gamma}^2 z_1^2}{y^2(y^4 + \tilde{\gamma}^2 z_1^2 (y^2 + z_1^2 + z_2^2))}\right)(y dy + z_1 dz_1 + z_2 dz_2)^2\,, \end{aligned} \tag{3.24}$$

where

$$G = \frac{y^4}{y^4 + \tilde{\gamma}^2 z_1^2 (y^2 + z_1^2 + z_2^2)}\,. \tag{3.25}$$

Let us now consider various limits of this geometry. As $y \to \infty$, G tends to unity, and one recovers the original undeformed geometry. This corresponds to an infrared limit in the dual theory and, as one would expect, the physics should become insensitive in this limit to a non-commutativity parameter [31, 32]. In the $y \to 0$ limit G scales according to

$$G \sim \frac{y^4}{\tilde{\gamma}^2 z_1^2 (z_1^2 + z_2^2)}\,, \qquad (y \to 0)\,, \tag{3.26}$$

so that one obtains

$$ds^2/R^2 = \frac{1}{y^2}\left(dy^2 + dz_1^2 + dz_2^2\right) - \left(\frac{1}{y^2(z_1^2 + z_2^2)}\right)(z_1 dz_1 + z_2 dz_2)^2 - \frac{1}{y^2} z_2^2 d\varphi_3^2\,. \tag{3.27}$$

This corresponds to a UV limit, where we expect the dual theory to acquire modifications depending on the non-commutativity parameter.

It is worthwhile to note that the curvature invariants possess a certain scaling symmetry. In the $y \to 0$ limit, the metric in eq. (3.27) is symmetric under $y \to \lambda y$, $(z_1, z_2) \to (\lambda z_1, \lambda z_2)$. The curvature invariants therefore do not depend on y as $y \to 0$, and thus remain bounded in this limit.

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While this geometry exhibits many of the properties known to persist in gravity duals of non-commutative Yang-Mills theories, there are some unexpected features as well. If we consider the subspace defined by $z_1 = 0$, we see that the $\tilde{\gamma}$ dependence drops out of the metric entirely. Furthermore, if we look at the scale where the deformation is noticeable, we find that $y \sim 1/\tilde{\gamma} z_1$. The theory therefore appears to have a position-dependent non-commutativity scale. Non-commutative gauge theories with a non-constant parameter have been studied in the context of the AdS/CFT correspondence [39 – 41], where they arise from non-vanishing NS-NS H flux and give rise to a non-associative star product [42]. To be certain, it is useful to point out that we have allowed deformations to act on isometries corresponding to the Cartan generators of the four-dimensional conformal group. This should be contrasted with the studies in [31, 32], where deformations were chosen to act on isometries on the boundary in a Poincaré patch [30]. Although our focus here is primarily restricted to the stringy aspects of the correspondence, these considerations clearly present a number of interesting issues on the field theory side that remain to be explored.

3.3 A note on finite temperature

From the point of view of the deformed Euclidean AdS_5 metric in eq. (3.24), it is not obvious what the finite-temperature extension should be. To study this question, we instead start from TsT deformations of the standard finite-temperature gravity solution.

The non-extremal Euclideanized D3 brane background with temperature T is given by

$$ds^2/R^2 = u^2 \left(h(u)\, d\hat{t}^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + \frac{du^2}{h(u)u^2} + d\Omega_5^2 \,, \tag{3.28}$$

where $h(u) \equiv 1 - u_0^4/u^4$, $u_0 = \pi T$ and $d\Omega_5^2$ is the metric on S^5. (Here $\hat{t}$ is understood to be compact.) Defining z_1 and $\hat{\varphi}_2$ so that

$$dx_1^2 + dx_2^2 = dz_1^2 + z_1^2 d\hat{\varphi}_2^2 \,, \tag{3.29}$$

we can perform a TsT transformation on the global $\mathrm{U}(1) \times \mathrm{U}(1)$ isometry parameterized by $\hat{\varphi}_2$ and $\hat{t}$. The resulting γ-deformed metric is

$$ds_{T\neq 0}^2/R^2 = u^2 \left(h(u) G\, dt^2 + dz_1^2 G\, d\varphi_2^2 + dx_3^2\right) + \frac{du^2}{u^2 h(u)} + d\Omega_5^2 \,, \tag{3.30}$$

where

$$G = \frac{1}{1 + \tilde{\gamma}^2 u^4 h(u) z_1^2} \,. \tag{3.31}$$

This metric, though obviously different from (3.27), is somewhat similar to the zero-temperature case. One salient feature is that the deformation vanishes at the horizon, since $G = 1$ when $h(u = u_0) = 0$. This fact ensures that thermodynamic properties are not spoiled. In particular, the eight-dimensional area of the horizon is unchanged, and the surface gravity

$$\kappa = -\frac{1}{2}(\nabla^a \chi^b)(\nabla_a \chi_b) \,, \tag{3.32}$$

where $\chi = \frac{\partial}{\partial t}$, is also unaffected by the deformation. By the first law, the energy is also unchanged, though this could be checked directly by calculating the mass in this deformed background. The total number of degrees of freedom is therefore preserved under the deformation. We expect that the same holds for the other deformations of interest: in the case of the TsT-deformed S^5 background, this statement corresponds to the total volume being unchanged by the deformation.

4. Classical integrability

To demonstrate that string theory on our γ-deformed AdS_5 background remains integrable at the classical level, we formulate the theory in terms of a Lax representation. As argued by Frolov in [15], the crucial issue lies in finding a local Lax pair invariant under the U(1) isometry transformations that are generated as part of the TsT deformation. Such a representation was found for the γ-deformed S^5 system in [15], and, in describing the analogous computation on the deformed AdS_5 geometry, we will closely follow the treatment therein.

A useful parameterization of the bosonic coset space $(\mathrm{SO}(4,2) \times \mathrm{SO}(6))/(\mathrm{SO}(5,1) \times \mathrm{SO}(5))$ was given in [43], where the the AdS_5 sector takes the form

$$g = \begin{pmatrix} 0 & Z_1 & -Z_3 & \bar{Z}_2 \\ -Z_1 & 0 & Z_2 & \bar{Z}_3 \\ Z_3 & -Z_2 & 0 & -\bar{Z}_1 \\ -\bar{Z}_2 & -\bar{Z}_3 & \bar{Z}_1 & 0 \end{pmatrix}, \qquad Z_i \equiv \eta_i e^{i\hat{\varphi}_i} . \tag{4.1}$$

With the condition in eq. (3.4), this matrix is constructed to satisfy

$$g^\dagger \, s \, g = s \,, \qquad s \equiv \mathrm{diag}(-1,-1,1,1) \,. \tag{4.2}$$

In other words, g is an $\mathrm{SU}(2,2)$ embedding of an element of the coset $\mathrm{SO}(4,2)/\,\mathrm{SO}(5,1)$. The action for the sigma model is then given by that of the principal chiral model

$$S = \int d\tau d\sigma \gamma^{\alpha\beta} \mathrm{Tr}\left(g^{-1}\partial_\alpha g \, g^{-1}\partial_\beta g \right) \,. \tag{4.3}$$

The Lax formulation of an integrable system encodes the (typically nonlinear) equations of motion in an auxiliary linear problem, which, in turn, is defined as a flatness condition on conserved currents. The Lax operator D_α for the sigma model can be written (as a function of a spectral parameter x) as

$$D_\alpha = \partial_\alpha - \frac{j^+_\alpha}{2(x-1)} + \frac{j^-_\alpha}{2(x+1)} \equiv \partial_\alpha - \mathcal{A}_\alpha(x) \,, \tag{4.4}$$

where j^+_α and j^-_α are self-dual and anti-self-dual projections of the right current

$$j_\alpha = g^{-1}\partial_\alpha g \,, \tag{4.5}$$

and $\mathcal{A}_\alpha(x)$ is the (right) Lax connection. The equations of motion ($\partial_\alpha(\gamma^{\alpha\beta} j_\beta) = 0$) are thus encoded by the condition[3]

$$[D_\alpha, D_\beta] = 0 \,. \tag{4.6}$$

[3]To be sure, the auxiliary linear problem is a system of equations for which eq. (4.6) stands as a consistency condition.

As noted, the goal is to find a Lax representation invariant under the TsT deformation that leads to the γ-deformed AdS_5 geometry computed in eq. (3.6). As it stands, the Lax operator in eq. (4.4) exhibits an explicit dependence on the U(1) coordinates $\hat{\varphi}_i$. Following [15], it is straightforward to demonstrate this dependence by noting the factorization

$$g(\eta_i, \hat{\varphi}_i) = M(\hat{\varphi}_i)\tilde{g}(\eta_i)M(\hat{\varphi}_i)\,, \qquad M(\hat{\varphi}_i) = e^{\Phi(\hat{\varphi}_i)}\,, \tag{4.7}$$

where the matrix Φ is given by

$$\Phi(\hat{\varphi}_i) = \frac{i}{2}\begin{pmatrix} \hat{\varphi}_1 - \hat{\varphi}_2 + \hat{\varphi}_3 & 0 & 0 & 0 \\ 0 & \hat{\varphi}_1 + \hat{\varphi}_2 - \hat{\varphi}_3 & 0 & 0 \\ 0 & 0 & -\hat{\varphi}_1 + \hat{\varphi}_2 + \hat{\varphi}_3 & 0 \\ 0 & 0 & 0 & -\hat{\varphi}_1 - \hat{\varphi}_2 - \hat{\varphi}_3 \end{pmatrix}, \tag{4.8}$$

and the η_i-dependent matrix $\tilde{g}(\eta_i)$ takes the form

$$\tilde{g}(\eta_i) = \begin{pmatrix} 0 & \eta_1 & -\eta_3 & \eta_2 \\ -\eta_1 & 0 & \eta_2 & \eta_3 \\ \eta_3 & -\eta_2 & 0 & -\eta_1 \\ -\eta_2 & -\eta_3 & \eta_1 & 0 \end{pmatrix}. \tag{4.9}$$

At this point it is easy to see that the non-derivative dependence of the Lax current j_α on the U(1) coordinates $\hat{\varphi}_i$ can be gauged away:

$$\tilde{j}_\alpha(\eta_i, \partial\hat{\varphi}_i) = M j_\alpha(\eta_i, \hat{\varphi}_i) M^{-1}\,. \tag{4.10}$$

This yields the following explicit form:

$$\tilde{j}_\alpha = \tilde{g}^{-1}\partial_\alpha \tilde{g} + \tilde{g}^{-1}\partial_\alpha \Phi \tilde{g} + \partial_\alpha \Phi\,. \tag{4.11}$$

Under this gauge transformation we obtain a suitable Lax operator, local under the γ deformation described above.[4]

4.1 $\mathfrak{sl}(2)_\gamma$ Lax representation

One remarkable property of the TsT transformations on the S^5 subspace is that both the gauge and string theories neatly exhibit γ-deformed analogues of the classically closed $\mathfrak{su}(2)$ subsector. This is auspicious, as the myriad techniques associated with the closure of this sector that have proved useful in the undeformed case come to bear in the deformed theory. Here we are interested in the analogous truncation from the full sigma model on AdS_5 to a closed $\mathfrak{sl}(2)$ sector (which amounts to a geometrical reduction to $AdS_3 \times S^1$). It turns out that the γ-deformed AdS_5 theory indeed admits the same consistent truncation to a deformed $\mathfrak{sl}(2)_\gamma$ subsector.

A convenient coordinate parameterization in this sector is given by the following SL(2) matrix [20]:

$$g = \begin{pmatrix} \cos\hat{\varphi}_1 \cosh\rho + \cos\hat{\varphi}_2 \sinh\rho & \sin\hat{\varphi}_1 \cosh\rho - \sin\hat{\varphi}_2 \sinh\rho \\ -\sin\hat{\varphi}_1 \cosh\rho - \sin\hat{\varphi}_2 \sinh\rho & \cos\hat{\varphi}_1 \cosh\rho - \cos\hat{\varphi}_2 \sinh\rho \end{pmatrix}. \tag{4.12}$$

[4] For further details, the reader is referred to [15, 9].

Here again, one is faced with the problem of defining a local Lax current invariant under γ deformations: the above parameterization clearly exhibits an explicit linear dependence on the U(1) coordinates $\hat{\varphi}_i$. A suitable gauge transformation, analogous to that in eq. (4.10), can be found by rewriting eq. (4.12) as

$$g = e^{\frac{i}{2}(\hat{\varphi}_1+\hat{\varphi}_2)\sigma_2} e^{\rho\sigma_3} e^{\frac{i}{2}(\hat{\varphi}_1-\hat{\varphi}_2)\sigma_2}\,, \tag{4.13}$$

where σ_i are the usual Pauli matrices. By assigning $M = e^{\frac{i}{2}(\hat{\varphi}_1-\hat{\varphi}_2)\sigma_2}$, we obtain the gauge-transformed right current:

$$\begin{aligned} \tilde{\jmath}_\alpha(\eta_i, \partial\hat{\varphi}_i) &= M j_\alpha(\eta_i, \hat{\varphi}_i) M^{-1} \\ &= \begin{pmatrix} \partial_\alpha\rho & e^{-\rho}(\partial_\alpha\hat{\varphi}_1\cosh\rho - \partial_\alpha\hat{\varphi}_2\sinh\rho) \\ -e^{-\rho}(\partial_\alpha\hat{\varphi}_1\cosh\rho + \partial_\alpha\hat{\varphi}_2\sinh\rho) & -\partial_\alpha\rho \end{pmatrix}. \end{aligned} \tag{4.14}$$

Invoking the same gauge transformation on the operator D_α yields

$$D_\alpha \to M D_\alpha M^{-1} \equiv \partial_\alpha - \mathcal{R}_\alpha\,, \tag{4.15}$$

from which one obtains a gauge-transformed Lax connection:

$$\mathcal{R}_\alpha = M\mathcal{A}_\alpha M^{-1} - M\partial_\alpha M^{-1} = \tilde{\mathcal{A}}_\alpha + \frac{i}{2}(\partial_\alpha\hat{\varphi}_1 - \partial_\alpha\hat{\varphi}_2)\sigma_2\,. \tag{4.16}$$

The auxiliary linear problem defined by the Lax formulation introduces a monodromy

$$\Omega(x) = \mathcal{P}\exp\int_0^{2\pi} d\sigma\,\mathcal{R}_1(x)\,, \tag{4.17}$$

and the usual quasi-momentum $p(x)$ is defined in terms of this monodromy as

$$\operatorname{Tr}\Omega(x) = 2\cos p(x)\,. \tag{4.18}$$

According to the standard argument, the quasi-momentum is conserved (i.e., does not depend on τ) because the trace of the holonomy of a flat connection does not depend on the contour of integration. The dependence of $p(x)$ on the spectral parameter x then implies an infinite set of conserved integrals of motion (which may be obtained, for example, by Taylor expansion in x).

4.2 Thermodynamic Bethe equations

With a suitable Lax representation in hand, we can encode the spectral problem in a thermodynamic Bethe equation by studying the pole structure and asymptotics of the quasi-momentum $p(x)$ on the complex x plane. Here the Bethe ansatz appears as a Riemann-Hilbert problem. These techniques were developed for the $\mathfrak{su}(2)$ sector of the gauge theory in [19], and later applied to the $\mathfrak{sl}(2)$ sector in [20]. We will follow these treatments closely, employing the methods presented in [9] for dealing with the general problem of including γ

deformations. In this respect, it turns out to be advantageous to work with both the gauge-transformed Lax connection $\mathcal{R}_\alpha$ in eq. (4.16) and the original connection $\mathcal{A}_\alpha$ appearing in in eq. (4.4).

To fix boundary conditions in the Riemann-Hilbert problem, one must first study the structure of the quasi-momentum $p(x)$ in the deformed theory. Following the analogous argument in [9], we note that the Lax connections in both the deformed and undeformed sigma models can be simultaneously diagonalized by a gauge transformation depending only on $\partial_\alpha \hat{\varphi}_i$ (and hence invariant under the deformation). This implies that the poles developed by the quasi-momentum at $x = \pm 1$ will be the same in both theories:

$$p(x) = \pi \frac{J/\sqrt{\lambda} \mp m}{x \pm 1} + \cdots \qquad x \to \mp 1\,, \tag{4.19}$$

where m is the winding number around the decoupled S^1 in $AdS_3 \times S^1$ (classical strings in the $\mathfrak{sl}(2)$ subsector propagate on this subspace).

At $x = 0$ and $x = \infty$ the quasi-momentum develops additional constant contributions due to the deformation. In these cases it is helpful to rely on the asymptotics of the original Lax connection $\mathcal{A}_\alpha$ by invoking an inverse gauge transformation on the monodromy [9]:

$$T(x) = M(2\pi) \mathcal{P} \exp \int_0^{2\pi} d\sigma \mathcal{A}_1(x) M^{-1}(0)\,. \tag{4.20}$$

This yields the following representation of the quasi-momentum:

$$2\cos p(x) \;=\; \mathrm{Tr}\; M_R\, \mathcal{P} \exp \int_0^{2\pi} d\sigma \mathcal{A}_1(x)\,, \tag{4.21}$$

$$\mathcal{A}_1(x) \;=\; \frac{j_1}{x^2-1} + \frac{x\, j_0}{x^2-1}\,, \tag{4.22}$$

where the matrix M_R is given by

$$M_R = M^{-1}(0) M(2\pi) = \begin{pmatrix} \cos\gamma\pi(S-\Delta) & -\sin\gamma\pi(S-\Delta) \\ \sin\gamma\pi(S-\Delta) & \cos\gamma\pi(S-\Delta) \end{pmatrix}. \tag{4.23}$$

We have made use here of the twisted boundary conditions in eq. (3.17) above, where we identify the U(1) charges J_1 and J_2 with the energy Δ and impurity number S of corresponding string energy eigenstates according to

$$J_1 = -\Delta\,, \qquad J_2 = S\,. \tag{4.24}$$

This notation is borrowed from the undeformed $\mathfrak{sl}(2)$ sector, where Δ and S are respectively mapped to the dimension and spin of corresponding $\mathfrak{sl}(2)$ operators in the gauge theory.

By parameterizing the right and left currents of the sigma model in a standard fashion

$$j_\alpha = g^{-1}\partial_\alpha g = \frac{1}{2} j_\alpha \cdot \hat{\sigma}\,, \qquad l_\alpha = \partial_\alpha g g^{-1} = \frac{1}{2} l_\alpha \cdot \hat{\sigma}\,, \tag{4.25}$$

where $\hat{\sigma} = (i\sigma_2, \sigma_3, -\sigma_1)$, one may refer directly to the undeformed $\mathfrak{sl}(2)$ problem [20] to compute

$$\frac{\sqrt{\lambda}}{4\pi} \int_0^{2\pi} d\sigma\, j_0^0 = \Delta + S\,, \qquad \frac{\sqrt{\lambda}}{4\pi} \int_0^{2\pi} d\sigma\, l_0^0 = \Delta - S\,. \tag{4.26}$$

The asymptotics of the quasi-momentum at $x = 0$ and $x = \infty$ can thus be found by expanding eq. (4.21) in the spectral parameter and following the prescription provided in [9] for discarding nonlocal contributions.[5] We find

$$
\begin{aligned}
p(x) &= \pi\gamma(\Delta - S) + 2\pi\frac{\Delta + S}{\sqrt{\lambda}\,x} + \cdots\,, \qquad x \to \infty\,, \\
p(x) &= \pi\gamma(\Delta + S) - 2\pi\frac{\Delta - S}{\sqrt{\lambda}}x + \cdots\,, \qquad x \to 0\,.
\end{aligned}
\tag{4.27}
$$

A resolvent function analytic on the complex x plane can therefore be defined by subtracting from the quasi-momentum its poles at $x = \pm 1$ and the constant contribution $\pi\gamma(\Delta - S)$ at $x \to \infty$:

$$
G(x) = p(x) - \pi\frac{J/\sqrt{\lambda} + m}{x - 1} - \pi\frac{J/\sqrt{\lambda} - m}{x + 1} - \pi\gamma(\Delta - S)\,. \tag{4.28}
$$

(Subtracting the constant piece serves to allow sensible Cauchy integrals over the spectral density, defined below.) From eq. (4.27), one obtains the following asymptotic behavior

$$
\begin{aligned}
G(x) &= \frac{2\pi}{\sqrt{\lambda}\,x}(\Delta + S - J) + \cdots\,, \qquad x \to \infty \\
G(x) &= 2\pi(m + \gamma S) + \frac{2\pi x}{\sqrt{\lambda}}(S - \Delta + J) + \cdots\,, \qquad x \to 0\,.
\end{aligned}
\tag{4.29}
$$

The usual spectral representation of the resolvent takes the form

$$
G(x) = \int_C dx'\,\frac{\sigma(x')}{x - x'}\,, \qquad C = C_1 \cup C_2 \ldots \cup C_n\,, \tag{4.30}
$$

where $\sigma(x)$ stands as a spectral density function supported on a finite number of cuts C_i in the complex x plane. Given that (by construction) $G(x)$ is analytic in the spectral parameter, it is straightforward to derive the following constraint equations on $\sigma(x)$:

$$
\begin{aligned}
\int_C dx\,\sigma(x) &= \frac{2\pi}{\sqrt{\lambda}}(\Delta + S - J)\,, \\
\int_C dx\,\frac{\sigma(x)}{x} &= -2\pi(m + \gamma S)\,, \\
\int_C dx\,\frac{\sigma(x)}{x^2} &= \frac{2\pi}{\sqrt{\lambda}}(\Delta - S - J)\,.
\end{aligned}
\tag{4.31}
$$

Unimodularity of the monodromy $\Omega(x)$ implies

$$
p(x + i0) + p(x - i0) = 2\pi n_k\,, \qquad x \in C_k\,, \tag{4.32}
$$

where the mode integer n_k labels the set of eigenvalues supported in the k^{th} contour C_k: this number corresponds to the mode number of individual impurity excitations on the string worldsheet. One thereby obtains the finite-gap integral equation

$$
2\fint_C dx'\,\frac{\sigma(x')}{x - x'} = -2\pi\left(\frac{J/\sqrt{\lambda} + m}{x - 1} + \frac{J/\sqrt{\lambda} - m}{x + 1}\right) + 2\pi n_k - 2\pi\gamma(\Delta - S)\,, \tag{4.33}
$$

[5] Adopting the methodology in [9] appears to generate sensible answers for the $\mathfrak{sl}(2)_\gamma$ sector, despite the possible ambiguities noted therein.

where, as usual, x is understood in this context to take values in C_k. This equation represents a thermodynamic Bethe ansatz in the classical limit of the deformed string theory in the $\mathfrak{sl}(2)_\gamma$ sector. As expected, the $\gamma \to 0$ limit of this equation reduces to the original Riemann-Hilbert problem obtained for the undeformed $\mathfrak{sl}(2)$ sector in [20].

Compared to the corresponding result in the deformed $\mathfrak{su}(2)_\gamma$ sector, we have obtained a relatively complicated modification to the undeformed $\mathfrak{sl}(2)$ thermodynamic Bethe equation. Using the constraints in eqs. (4.31), we can rewrite eq. (4.33) in a slightly more illuminating form:

$$
\begin{aligned}
2\pi(n_k - \gamma J) - 4\pi \frac{x\, J/\sqrt{\lambda}}{x^2-1} \;&=\\
&2 \fint_C dx'\, \sigma(x') \left(\frac{1}{x-x'} - \frac{2x' + \gamma\sqrt{\lambda}\,(x'^2-1)}{2\,x'^2(x^2-1)} + \frac{\gamma\sqrt{\lambda}}{2}\,\frac{1}{x'^2} \right) . \qquad (4.34)
\end{aligned}
$$

In addition to a global shift in the mode number n_k, several γ-dependent deformation terms now appear under the spectral integral. An obvious question at this point is whether the established technology for promoting thermodynamic Bethe equations to *discrete*, (or "quantum") Bethe equations will be completely reliable for string theory in this wider class of γ-deformed backgrounds. In the $\mathfrak{su}(2)_\gamma$ sector, the TsT deformation simply amounts to a shift by γJ in the mode number n_k: the $\mathfrak{su}(2)_\gamma$ problem is therefore solved trivially by invoking a corresponding shift in the known $\mathfrak{su}(2)$ quantum string Bethe ansatz. While we have presently obtained the same shift in the $\mathfrak{sl}(2)_\gamma$ sector of the string theory, we have also generated additional γ-dependent contributions on the right-hand side of eq. (4.34), which mark an interesting nontrivial deformation of the problem. We will return to the crucial issue of deformed quantum string Bethe ansätze in section 6.

4.3 General deformations

To some extent we have circumvented the problem of T-duality along timelike directions. We are forced to confront this issue, however, if we wish to consider a wider class of deformations achieved by sequential TsT transformations on all of the U(1) angular coordinates $\hat{\varphi}_i$ in AdS_5. It turns out that, with respect to the goals set forth in the present study, it is efficient to adopt a pragmatic viewpoint by invoking deformations in a strictly formal manner. If one prefers, the resulting background may then be studied at face-value, and not necessarily as a deformation of any particular parent geometry.

Following [15], we obtain a three-parameter deformation of AdS_5 using a chain of TsT transformations on each of the three tori in AdS_5 parameterized by the undeformed U(1) angular coordinates $(\hat{\varphi}_1, \hat{\varphi}_2)$, $(\hat{\varphi}_2, \hat{\varphi}_3)$ and $(\hat{\varphi}_1, \hat{\varphi}_3)$:

$$
\begin{aligned}
ds^2/R^2 &= g^{ij} d\eta_i d\eta_j + G(-\eta_1^2 d\varphi_1^2 + \eta_2^2 d\varphi_2^2 + \eta_3^2 d\varphi_3^2) - G\eta_1^2\eta_2^2\eta_3^2 \big[d(\sum_{i=1}^{3} \tilde{\gamma}_i \varphi_i)\big]^2 ,\\
B_2 &= R^2 G w_2 ,\\
w_2 &\equiv -\tilde{\gamma}_3\, \eta_1^2\eta_2^2\, d\varphi_1 \wedge d\varphi_2 + \tilde{\gamma}_1\, \eta_2^2\eta_3^2\, d\varphi_2 \wedge d\varphi_3 - \tilde{\gamma}_2\, \eta_3^2\eta_1^2\, d\varphi_3 \wedge d\varphi_1 , \qquad (4.35)
\end{aligned}
$$

with

$$G^{-1} \equiv 1 - \tilde{\gamma}_3^2\, \eta_1^2\eta_2^2 + \tilde{\gamma}_1^2\, \eta_2^2\eta_3^2 - \tilde{\gamma}_2^2\, \eta_1^2\eta_3^2 \,. \tag{4.36}$$

Starting from this geometry, one may proceed according to the methodology described above. As with the $\mathfrak{su}(2)_\gamma$ sectors described in section 2, we find that each individual $\mathfrak{sl}(2)_\gamma$ sector, to $O(1/R^2)$ in the large-radius expansion, is deformed by a single element of the set $\{\tilde{\gamma}_i\}$. Following section 3, one obtains the following transformation conditions under the full sequence of TsT deformations, analogous to eq. (3.15) above:

$$\begin{aligned}
\hat{\varphi}_1' &\to \varphi_1' - \gamma_3\, p_2 - \gamma_2\, p_3 \,, \\
\hat{\varphi}_2' &\to \varphi_2' + \gamma_1\, p_3 + \gamma_3\, p_1 \,, \\
\hat{\varphi}_3' &\to \varphi_3' + \gamma_2\, p_1 - \gamma_1\, p_2 \,.
\end{aligned} \tag{4.37}$$

(We again introduce the modified deformation parameters $\gamma_i = \tilde{\gamma}_i/\sqrt{\lambda}$.) We note that the sign of γ_3 is a consequence of the $\hat{\varphi}_2$, $\hat{\varphi}_1$, $\hat{\varphi}_2$ ordering of the corresponding TsT transformation; this sign would be reversed if we instead chose the sequence $\hat{\varphi}_1$, $\hat{\varphi}_2$, $\hat{\varphi}_1$ (the other transformations are ordered as in [15]: $\hat{\varphi}_2$, $\hat{\varphi}_3$, $\hat{\varphi}_2$ and $\hat{\varphi}_1$, $\hat{\varphi}_3$, $\hat{\varphi}_1$). We therefore obtain the following twisted boundary conditions on the undeformed U(1) coordinates $\hat{\varphi}_i$ that arise under this chain of TsT deformations (again, m_i stand for winding numbers):

$$\begin{aligned}
\hat{\varphi}_1(2\pi) - \hat{\varphi}_1(0) &= 2\pi(m_1 - \gamma_3\, J_2 - \gamma_2\, J_3)\,, \\
\hat{\varphi}_2(2\pi) - \hat{\varphi}_2(0) &= 2\pi(m_2 + \gamma_1\, J_3 + \gamma_3\, J_1)\,, \\
\hat{\varphi}_3(2\pi) - \hat{\varphi}_3(0) &= 2\pi(m_3 + \gamma_2\, J_1 - \gamma_1\, J_2)\,.
\end{aligned} \tag{4.38}$$

With these boundary conditions, it is easy to follow the above procedures to obtain a Lax representation and thermodynamic Bethe ansatz in this more general deformed geometry. Since truncation to $\mathfrak{sl}(2)_\gamma$ subsectors restricts to a single γ_i deformation, however, questions pertaining to these sectors can be studied by choosing a single TsT transformation. (In the coordinate system employed in this paper, the deformation parameterized by γ_1 is in fact trivial upon truncation to an $\mathfrak{sl}(2)_\gamma$ subsystem.)

5. Twisted string spectra in the near-pp-wave limit

We now turn to the task of gathering data on the spectrum of string states in protected sectors of the string theory on γ-deformed $AdS_5 \times S^5$. For arbitrary numbers of worldsheet excitations, with arbitrary subsets of bound states (marked by subsets of confluent mode numbers), formulas for the perturbative $O(1/J)$ energy correction away from the pp-wave limit are typically complicated, and provide a fairly rigorous test of any conjectural Bethe equations that purport to encode such spectral information [44]. In this section we will compute these near-pp-wave corrections in the deformed $\mathfrak{su}(2)_\gamma$ and $\mathfrak{sl}(2)_\gamma$ sectors described above.

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5.1 $\mathfrak{su}(2)_\gamma$ sector

Since we are working within bosonic truncations of the full superstring theory on the γ-deformed geometry, we find it convenient to approach the computation of energy spectra using a purely Hamiltonian formalism.[6] One particular advantage of such a framework is that components of the worldsheet metric may be employed as Lagrange multipliers enforcing the Virasoro constraints. As such, we need not compute curvature corrections to the worldsheet metric that are inevitable in other approaches. Omitting the computational details, we find the following lightcone Hamiltonian, truncated to an $\mathfrak{su}(2)_\gamma$ sector on the deformed S^5 subspace:

$$H_{\rm LC} = \frac{1}{G^{++}p_-}\left(G^{+-}\tilde{p}_- p_- - \sqrt{F}\right) - \frac{p_A x'^A}{p_-}B_{+-} + B_{+y}y' + B_{+\bar{y}}\bar{y}' \,. \tag{5.1}$$

For the sake of compactness, we have defined the quantity

$$\begin{aligned} F \equiv (G^{+-})^2\tilde{p}_-^2 p_-^2 - G^{++}\Big[G_{--}(p_A x'^A)^2 + p_-^2\Big(\tilde{p}_y G^{yy}\tilde{p}_y + 2\tilde{p}_y G^{y\bar{y}}\tilde{p}_{\bar{y}} + \tilde{p}_{\bar{y}} G^{\bar{y}\bar{y}}\tilde{p}_{\bar{y}} \\ +G^{--}\,\tilde{p}_-^2 + y' G_{yy} y' + 2y' G_{y\bar{y}}\bar{y}' + \bar{y}' G_{\bar{y}\bar{y}}\bar{y}'\Big)\Big] \,. \end{aligned} \tag{5.2}$$

The notation $\tilde{p}_-$ and $\tilde{p}_y$ is used to indicate corrections to the usual conjugate momenta p_- and p_y due to the presence of a nonzero B-field:

$$\begin{aligned} \tilde{p}_- &\equiv p_- + B_{-y}y' + B_{-\bar{y}}\bar{y}' \,, \\ \tilde{p}_y &\equiv p_y + B_{y\bar{y}}\bar{y}' + B_{-y}\frac{p_A x'^A}{p_-} \,. \end{aligned} \tag{5.3}$$

The vector x^A is understood to span the coordinate set $(x^+, x^-, y, \bar{y})$, where the complex fields y and $\bar{y}$ were defined in eq. (2.8) above. The restriction to the complex pair $(y, \bar{y})$ (as opposed to the $(z, \bar{z})$ pair from eq. (2.8)) corresponds to the truncation from the full theory on the γ-deformed S^5 to an $\mathfrak{su}(2)_\gamma$ sector.[7] In the corresponding $\mathfrak{sl}(2)_\gamma$ truncation on the deformed AdS_5 subspace we will project onto coordinates $(x^+, x^-, v, \bar{v})$, where v and $\bar{v}$ are defined in eq. (3.12).

We aim to compute near-pp-wave energy spectra in a semiclassical expansion about point-like (or BMN) string solutions. (Note that analogous corrections were found for a different set of string solutions in an $\mathfrak{su}(2)_\gamma$ sector in [9].) Arranging the large-radius (equivalently, large-J) expansion of the lightcone $\mathfrak{su}(2)_\gamma$ Hamiltonian according to

$$H_{\rm LC} = H_0 + \frac{H_{\rm int}}{R^2} + O(1/R^4) \,, \tag{5.4}$$

[6] See, e.g., [45] for a detailed description of such an approach in the undeformed $AdS_5 \times S^5$ background.

[7] Technically, the theory restricted to $(y, \bar{y})$ coordinates describes a system that is slightly larger than the closed $\mathfrak{su}(2)_\gamma$ sector. To fully truncate to $\mathfrak{su}(2)_\gamma$ we will perform an additional projection that is described below.

we obtain the following as functions of coordinate fields on the γ-deformed S^5 subspace:

$$\begin{aligned}
H_0(S^5_{\tilde\gamma}) &= \frac{1}{2p_-}\Big[4|p_y|^2+|y'|^2-ip_-(y'\bar y-y\bar y')\tilde\gamma+p_-^2|y|^2(1+\tilde\gamma^2)\Big]\,,\\
H_{\rm int}(S^5_{\tilde\gamma}) &= \frac{1}{8p_-^3}\Big\{-4p_y^2(4\bar p_y^2+p_-^2y^2-y'^2)-16p_-^2|p_y|^2|y|^2+p_-^2\bar y^2(3p_-^2y^2+y'^2-4\bar p_y^2)\\
&\quad+\bar y'^2(p_-^2y^2-y'^2+4\bar p_y^2)-2ip_-\tilde\gamma\Big[-4p_y^2yy'+p_-^2|y|^2(y\bar y'-\bar y y')\\
&\quad+\bar y'(4\bar p_y^2\bar y-y'^2\bar y+y|y'|^2)\Big]-p_-^2\tilde\gamma^2\Big[4p_y^2y^2+\bar y^2(4\bar p_y^2+2p_-^2y^2-y'^2)+4|y|^2|y'|^2\\
&\quad-y^2\bar y'^2+2ip_-\tilde\gamma|y|^2(y\bar y'-y'\bar y)+p_-^2\tilde\gamma^2|y|^4\Big]\Big\}\,. \qquad (5.5)
\end{aligned}$$

As expected, the pp-wave Hamiltonian is quadratic in worldsheet fluctuations, while the interaction Hamiltonian $H_{\rm int}$ appearing at $O(1/R^2)$ in the expansion contains terms that are uniformly quartic in fields.

The leading-order equations of motion are solved by the usual expansion in Fourier modes[8]

$$y(\tau,\sigma)=\sum_{n=-\infty}^{\infty}y_n(\tau)e^{-in\sigma}\,,\qquad p(\tau,\sigma)=\sum_{n=-\infty}^{\infty}p_n(\tau)e^{-in\sigma}\,, \qquad (5.6)$$

where

$$y_n(\tau)=\frac{i}{\sqrt{2\omega_n}}(a_ne^{-i\omega_n\tau}-\bar a^\dagger_{-n}e^{i\omega_n\tau})\,,\qquad p_n(\tau)=\frac{1}{2}\sqrt{\frac{\bar\omega_n}{2}}(\bar a_ne^{-i\bar\omega_n\tau}+a^\dagger_{-n}e^{i\bar\omega_n\tau})\,, \qquad (5.7)$$

and n denotes an integer mode index ($-\infty<n<\infty$). In the presence of a nonzero deformation, we obtain the following shifted dispersion relations:

$$\omega_n^2=p_-^2+(n-p_-\tilde\gamma)^2\,,\qquad \bar\omega_n^2=p_-^2+(n+p_-\tilde\gamma)^2\,. \qquad (5.8)$$

Upon expanding the interaction Hamiltonian in raising and lowering operators, we complete the projection onto the closed $\mathfrak{su}(2)_\gamma$ sector by keeping either the $(a_n, a^\dagger_{-n})$ or $(\bar a_n,\bar a^\dagger_{-n})$ oscillator pair and setting all remaining terms to zero. In essence, this achieves an SO(4) symmetric-traceless projection, in precise analogy with the undeformed theory [36, 37, 44, 46]. The Hamiltonian is then understood to block-diagonalize in the Fock subspace spanned by N-impurity string states composed of N raising operators in the $\mathfrak{su}(2)_\gamma$ projection acting on a ground state labeled by $|J\rangle$:

$$a^\dagger_{n_1}a^\dagger_{n_1}\cdots a^\dagger_{n_N}\,|J\rangle\,.$$

Following the usual conventions, it is convenient to replace the S^5 radius R with $\sqrt{p_-J}$ and arrange the large-J expansion of the energy spectrum according to

$$E(\{n_j\},J)=\sum_{j=1}^{N}\sqrt{1+(n_j\ -\tilde\gamma/\sqrt{\lambda'})^2\lambda'}+\delta E(\{n_j\},J)+O(1/J^2)\,. \qquad (5.9)$$

[8]Note that, strictly speaking, one is instructed in this formalism to compute equations of motion directly from the Hamiltonian rather than the Lagrangian. The difference between the two formalisms amounts to a sign flip on the deformation parameter $\tilde\gamma$.

The leading-order term in this expansion represents the familiar BMN energy formula with an additional $\tilde{\gamma}$-dependent shift in the mode index. The N worldsheet excitations can be labeled by N integer mode numbers n_j such that the full set $\{n_j\}$ is populated by M uniform subsets consisting of N_j equal mode numbers n_j ($j \in 1, \ldots, M$):[9]

$$\{n_j\} = \Big\{ \{\underbrace{n_1, n_1, \ldots, n_1}_{N_1}\}, \{\underbrace{n_2, n_2, \ldots, n_2}_{N_2}\}, \ldots, \{\underbrace{n_M, n_M, \ldots, n_M}_{N_M}\} \Big\}. \tag{5.10}$$

The perturbative energy shift in the near-pp-wave limit is then given by

$$\begin{aligned} \delta E_{\mathfrak{su}(2)_\gamma}(\{n_j\}, \{N_j\}, J) =& -\frac{1}{2J}\Bigg\{ \sum_{j=1}^{M} N_j(N_j - 1)\left[\left(1 + (\tilde{\gamma} - n_j\sqrt{\lambda'})^{-2}\right)^{-1}\right] \\ & - \sum_{\substack{j,k=1 \\ j\neq k}}^{M} \frac{N_j N_k}{\omega_{n_j}\omega_{n_k}\lambda'}\Big\{ -\lambda'(n_j n_k + n_k^2 + n_j^2(1 + n_k^2\lambda')) \\ & + \tilde{\gamma}((n_j + n_k)\sqrt{\lambda'} - \tilde{\gamma})(3 + 2n_j n_k\lambda' - (n_j + n_k)\sqrt{\lambda'}\tilde{\gamma} + \tilde{\gamma}^2) \\ & + \lambda'(n_j\sqrt{\lambda'} - \tilde{\gamma})(n_k\sqrt{\lambda'} - \tilde{\gamma})\omega_{n_j}\omega_{n_k}\Big\}\Bigg\}. \end{aligned} \tag{5.11}$$

At this point it is useful to recall (see eq. (3.16)) that the fixed parameter $\tilde{\gamma}$ appearing in the geometry is related to the corresponding deformation parameter in the β-deformed gauge theory by $\beta = \gamma = \tilde{\gamma}/\sqrt{\lambda}$. As noted in [9], this implies that the parameter

$$\tilde{\beta} \equiv \beta J\sqrt{\lambda'} \tag{5.12}$$

is also held fixed in the large-J expansion about the pp-wave limit. Since λ' is fixed and finite in this limit, βJ must also be held fixed. The above formula for the energy shift at $O(1/J)$ thus has a very simple interpretation: if we take the undeformed near-pp-wave energy correction [44]

$$\begin{aligned} \delta E_{\mathfrak{su}(2)}(\{n_j\}, \{N_j\}, J) =& -\frac{1}{2J}\Bigg\{ \sum_{j=1}^{M} N_j(N_j - 1)\left(1 - \frac{1}{\varpi_{n_j}^2\lambda'}\right) \\ & + \sum_{\substack{j,k=1 \\ j\neq k}}^{M} \frac{N_j N_k}{\varpi_{n_j}\varpi_{n_k}}\left[q_k^2 + q_j^2\varpi_{n_k}^2\lambda' + q_j q_k(1 - \varpi_{n_j}\varpi_{n_k}\lambda')\right]\Bigg\} \end{aligned} \tag{5.13}$$

(the symbol ϖ_n is specified by the undeformed dispersion relation $\varpi_n = \sqrt{p_-^2 + n^2}$), and shift the mode numbers by the fixed amount $n_j \to n_j - \beta J$, we obtain eq. (5.11) exactly. Based on observations made in [15, 9], this is precisely the outcome one should expect: the deformed theory is mapped from the original theory on S^5 by imposing twisted boundary conditions, analogous to those in eq. (3.17), on the relevant undeformed fields. In the $\mathfrak{su}(2)_\gamma$ sector, these boundary conditions only act to shift the mode numbers of the worldsheet excitations. We shall see a more dramatic modification in the $\mathfrak{sl}(2)_\gamma$ sector.

[9] Loosely speaking, one can think of each subset of N_j equal mode numbers as corresponding to a bound state on the string worldsheet. These states in turn fill out the support contours C_i in the complex plane of the spectral parameter x.

5.2 $\mathfrak{sl}(2)_\gamma$ sector

The near-pp-wave limit taken in the $\mathfrak{sl}(2)_\gamma$ sector yields the following string lightcone Hamiltonian, expanded to $O(1/J)$ near the pp-wave limit and projected onto the complex coordinate pair $(v, \bar{v})$:

$$\begin{aligned}
H_0(AdS_5^{\tilde{\gamma}}) &= \frac{1}{2p_-}\Big[4|p_v|^2 + |v'|^2 - ip_-(v'\bar{v} - v\bar{v}')\tilde{\gamma} + p_-^2|v|^2(1+\tilde{\gamma}^2)\Big]\,, \\
H_{\rm int}(AdS_5^{\tilde{\gamma}}) &= \frac{1}{8p_-^3}\Big\{16p_-^2|p_v|^2|v|^2 + (4\bar{p}_v^2 - v'^2)\bar{v}'^2 + 4ip_-^3|v|^2(v\bar{v}' - v'\bar{v})\tilde{\gamma}(1+\tilde{\gamma}^2) \\
&\quad + p_-^4|v|^4(-1 + 6\tilde{\gamma}^2 + 3\tilde{\gamma}^4) + 4p_v^2(-4\bar{p}_v^2 + v'^2 + p_-^2v^2(1+\tilde{\gamma}^2)) \\
&\quad + p_-^2\Big[4|v|^2|v'|^2\tilde{\gamma}^2 + 4\bar{p}_v^2\bar{v}^2(1+\tilde{\gamma}^2) - v'^2\bar{v}^2(1+\tilde{\gamma}^2) - v^2\bar{v}'^2(1+\tilde{\gamma}^2)\Big]\Big\}\,. \qquad (5.14)
\end{aligned}$$

Expanding in raising and lowering operators, we again project onto $\mathfrak{sl}(2)_\gamma$ sectors by setting either the $(a_n, a_{-n}^\dagger)$ or $(\bar{a}_n, \bar{a}_{-n}^\dagger)$ oscillator pair to zero. The perturbing Hamiltonian $H_{\rm int}(AdS_5^{\tilde{\gamma}})$ is then easily diagonalized in a corresponding basis of Fock states to yield the following energy shift at $O(1/J)$ in the near-pp-wave limit:

$$\begin{aligned}
\delta E_{\mathfrak{sl}(2)_\gamma}(\{n_j\},\{N_j\},J) &= \frac{1}{2J}\Bigg\{\sum_{j=1}^{M} N_j(N_j-1)\frac{(\tilde{\gamma} - n_j\sqrt{\lambda'})^2}{\omega_{n_j}^2\lambda'} \\
&\quad + \sum_{\substack{j,k=1\\ j\neq k}}^{M}\frac{N_jN_k}{\omega_{n_j}\omega_{n_k}\lambda'}\Big\{3\tilde{\gamma}^2 + \tilde{\gamma}^4 - (n_j+n_k)\tilde{\gamma}^3\sqrt{\lambda'} + n_jn_k\lambda'(1 - n_jn_k\lambda') \\
&\quad + (n_j+n_k)\tilde{\gamma}\sqrt{\lambda'}(n_jn_k\lambda' - 2) + \lambda'(n_kn_j\lambda' - \tilde{\gamma}^2)\omega_{n_j}\omega_{n_k}\Big\}\Bigg\}\,. \qquad (5.15)
\end{aligned}$$

In this sector the $O(1/J)$ energy spectrum is *not* related to the corresponding undeformed spectrum by a simple shift in the mode numbers. A closer inspection of eq. (5.15) reveals, however, that the deformed energy shift can be understood to arise from the combined effect of an overall shift in worldsheet mode numbers and an additional shift linear in the deformation parameter γ. Schematically, one finds that

$$\delta E_{\mathfrak{sl}(2)_\gamma}(\{n_j\}) = \delta E_{\mathfrak{sl}(2)}(\{n_j + \gamma J\}) + \gamma\,\delta E_2(\{n_j + \gamma J\})\,, \qquad (5.16)$$

where δE_2 is similar, but not identical, to the undeformed near-pp-wave energy shift $\delta E_{\mathfrak{sl}(2)}$ from [44]. In the next section we will study how these interesting modifications can be embedded in a discrete extension of the thermodynamic $\mathfrak{sl}(2)_\gamma$ Bethe ansatz computed in eqs. (4.33), (4.34).

6. Twisted quantum Bethe equations

As described above, the integral equations for the spectral density of the Lax operator in eqs. (4.33) and (4.34) can be interpreted as Bethe equations for the string theory in a classical, thermodynamic limit. This picture has led to a series of conjectures for how to

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discretize this limit of the "stringy" Bethe equations based on similarities with corresponding limits of the dual gauge theory [47 − 49]. One remarkable outcome is that the resulting discrete Bethe equations exactly reproduce near-pp-wave string energy spectra in various closed sectors of the theory. It is not entirely clear whether these techniques are specific to the duality connecting $\mathcal{N} = 4$ SYM theory with string theory on $AdS_5 \times S^5$. To pose the question more precisely, we ask if the discretization conjectures based on established gauge theory considerations can be applied with similar success in other contexts. Since the gauge theory side of the duality seems to be drastically modified under deformations corresponding (holographically) to TsT transformations on the AdS_5 subspace, the γ-deformed $\mathfrak{sl}(2)_\gamma$ sectors described here present an excellent opportunity to address such issues. In this section we will therefore apply familiar discretization techniques to the γ-deformed string theory on AdS_5 to derive a twisted quantum string Bethe ansatz for the spectral problem in these sectors. While we will briefly review the essential methodology, the reader is referred to [47 − 49] for further details.

The central conjecture is that the string theory spectrum is described by the diffractionless scattering of elementary excitations on the worldsheet [48]. In other words, the energy spectrum should be encoded in a fundamental equation in the excitation momenta p_k (and corresponding mode numbers n_k) of the form

$$p_k J = 2\pi n_k + \sum_{j\neq k} \theta(p_k, p_j)\,, \tag{6.1}$$

where the scattering phase $\theta(p_k, p_j)$ is defined in terms of a factorized S-matrix:[10]

$$\theta(p_k, p_j) = -i \log S(p_k, p_j)\,. \tag{6.2}$$

This two-body scattering matrix has been the locus of a great deal of deserved attention: the symmetry algebra of the theory constrains the form of the S matrix up to an overall phase [50], and it is suspected that this phase is further constrained by unitarity and crossing symmetry [51] (or certain worldsheet versions thereof). (For additional interesting developments, see [52, 53].) One certainly hopes that further insight into this problem will reveal a precise procedure by which the structure of the string theory S matrix might be uniquely determined. The second ingredient, which is key in the present scenario, is that the hidden local charges in the theory, labeled as Q_r, are expected to arise as linear sums over local dispersion relations $q_r(p_k)$:

$$Q_r = \sum_k q_r(p_k)\,. \tag{6.3}$$

Starting from the classical Bethe equations provided by the Lax representation of the string sigma model, discretized equations may be formulated by relying on cues provided by the gauge theory [47, 48]. One crucial test of this procedure is whether predictions obtained from the conjectured quantum Bethe equations match data coming directly from string theory computations. For example, the near-pp-wave energy shifts in the undeformed

[10]For present purposes we only display the bosonic version of this equation.

$\mathfrak{sl}(2)$ sector should be encoded in the (discretized) scattering phase $\theta(p_k, p_j)$ [48][11]

$$\delta\Delta(n_k, n_j, \gamma) = \lambda' \sum_{\substack{j,k=1 \\ j\neq k}}^{S} \frac{J}{2\pi} \frac{n_k}{\sqrt{1+\lambda' n_k^2}}\, \theta\left(2\pi n_k/J, 2\pi n_j/J\right) . \tag{6.4}$$

Amazingly, this general approach yields the correct energy spectrum in the undeformed theory at $O(1/J)$ for the closed $\mathfrak{su}(2)$, $\mathfrak{su}(1|1)$ and $\mathfrak{sl}(2)$ sectors, and for a variety of different string solutions (for details and further interesting developments, see [47 − 49, 54 − 59]).

As noted above, classical string Bethe equations were derived in the TsT-deformed $\mathfrak{su}(2)_\gamma$ sector in [9, 15]. As expected, these equations differed from those in the undeformed $\mathfrak{su}(2)$ sector by a simple global shift in the mode numbers $n_j,\ n_k$. The quantum extension of the $\mathfrak{su}(2)_\gamma$ Bethe equations is therefore rather simple, and the result manifestly agrees with the corresponding $\mathfrak{su}(2)_\gamma$ energy spectrum at $O(1/J)$, computed in eq. (5.11) above. (Recall that this energy shift may indeed be obtained from the undeformed spectrum by an overall shift in mode numbers.) An analogous treatment of the thermodynamic Bethe equations in the $\mathfrak{sl}(2)_\gamma$ sector requires a more careful analysis.

As demonstrated in [60, 48, 20], the detailed form of the thermodynamic string Bethe equations prevents one from adopting the naive interpretation of the function $\sigma(x)$, introduced in the spectral representation of the resolvent in eq. (4.30), as a density of string energy eigenstates supported on the contours C_i. This problem can be summarized by noting that the first condition in eq. (4.31)

$$\int_C dx\, \sigma(x) \sim \Delta + S - J \tag{6.5}$$

implies that the normalization of the spectral density is coupling-dependent, due to the presence of Δ on the right-hand side [20].

In [60, 20], it was shown that a legitimate excitation density ρ (with coupling-independent normalization) can be defined by introducing a nonlinear redefinition of the spectral parameter:

$$\varphi \equiv x + \frac{T}{x}\,, \tag{6.6}$$

where $T \equiv \lambda'/16\pi^2$, such that

$$\rho(\varphi) = \sigma(x)\,. \tag{6.7}$$

The quasi-momentum then depends on the spectral parameter φ according to

$$p(\varphi) = 1/\sqrt{\varphi^2 - 4T}\,. \tag{6.8}$$

[11] This particular equation holds for the $\mathfrak{sl}(2)$ sector only.

Starting from the continuum Bethe equation in the deformed $\mathfrak{sl}(2)_\gamma$ sector of the string theory (eq. (4.34)), we therefore invoke the change of variables in eq. (6.6) to obtain

$$
\begin{aligned}
2\fint d\varphi' \frac{\rho(\varphi')}{\varphi-\varphi'} &= 2\pi(n_k-\gamma J) - p(\varphi) \\
&+ \fint d\varphi' \rho(\varphi') \Bigg\{ \frac{2T}{\sqrt{\varphi'^2-4T}\sqrt{\varphi^2-4T}} \left(\frac{x}{T-xx'} - \frac{x'}{T-xx'} \right) \\
&\qquad + 4\pi\gamma J T \left(\frac{1}{x^2-T} - \frac{1}{x'^2-T} \right) \Bigg\} . \qquad (6.9)
\end{aligned}
$$

To keep this equation compact, x and x' are understood via eq. (6.6) to be functions of φ and φ', respectively.

Following [20, 60], we should be able to recast the integral term on the right-hand side of eq. (6.9) strictly in terms of the relativistic dispersion relations

$$
q_r(\varphi) = \frac{1}{\sqrt{\varphi^2-4T}} \left(\frac{1}{2}\varphi + \frac{1}{2}\sqrt{\varphi^2-4T} \right)^{1-r} . \qquad (6.10)
$$

The γ-independent terms are known to arise from an infinite sum over these $q_r(\varphi)$ [47]. We find that the remaining terms coming from the deformation can be written simply as a linear expression in the dispersion relation $q_2(\varphi)$:

$$
\begin{aligned}
2\fint d\varphi' \frac{\rho(\varphi')}{\varphi-\varphi'} &= 2\pi(n_k-\gamma J) - p(\varphi) \\
-2\fint d\varphi' \rho(\varphi') &\Bigg\{ \sum_{r=1}^{\infty} T^r \left(q_{r+1}(\varphi') q_r(\varphi) - q_r(\varphi') q_{r+1}(\varphi) \right) + 2\pi\gamma J T \left(q_2(\varphi) - q_2(\varphi') \right) \Bigg\} .
\end{aligned}
\qquad (6.11)
$$

Now, following [47, 48], we interpret this continuum equation as the thermodynamic limit of a discrete, or "quantum" Bethe ansatz. (To be sure, the thermodynamic limit is taken to render a distribution of Bethe roots that is macroscopic and smooth in φ: J and S become infinite, with the filling fraction S/J held fixed.) One is then instructed to rely on the conjectured all-loop Bethe equations in the dual gauge theory to properly discretize this equation. The final quantum string Bethe equation in the deformed $\mathfrak{sl}(2)_\gamma$ sector thus takes the form

$$
e^{i(p_k-2\pi\gamma)J} = \prod_{\substack{j=1\\ j\neq k}}^{S} \frac{\varphi(p_k)-\varphi(p_j)-i}{\varphi(p_k)-\varphi(p_j)+i} \, e^{-2\pi i\gamma g^2(q_2(p_k)-q_2(p_j))} \prod_{r=1}^{\infty} e^{-2i\theta_r(p_k,p_j)} , \qquad (6.12)
$$

where

$$
\theta_r(p_k,p_j) \equiv \left(\frac{g^2}{2} \right)^r \left(q_r(p_k) q_{r+1}(p_j) - q_{r+1}(p_k) q_r(p_j) \right) , \qquad (6.13)
$$

and, as usual, $g^2 \equiv \lambda/8\pi^2$. Remarkably, this equation precisely reproduces the $O(1/J)$ near-pp-wave energy shift computed above in eq. (5.15). Furthermore, it is easy to verify that the $\lambda^{1/4}$ strong-coupling behavior of the spectrum is not spoiled by the deformation.

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The dispersionless scattering of excitations in the worldsheet theory is therefore conjectured to be encoded in the following two-body scattering phase:

$$\theta(p_k, p_j, \gamma) \approx -\frac{2}{\varphi(p_k) - \varphi(p_j)} - 2\sum_{r=1}^{\infty} \theta_r(p_k, p_j) - 2\pi\gamma g^2 \left(q_2(p_k) - q_2(p_j)\right) . \qquad (6.14)$$

The γ-dependent deformation term in $\theta(p_k, p_j, \gamma)$ is determined by $q_2(p)$, which is the energy of a single excitation of momentum $q_1(p) = p$.[12] To make contact with the thermodynamic limit, one invokes the following rescaling:

$$p_k \to p_k/J\,, \qquad q_r(p_k) \to J^{-r} q_r(p_k)\,. \qquad (6.15)$$

It is then straightforward to see that eq. (6.11) is properly embedded in the discrete Bethe ansatz in eq. (6.12).

The nontrivial accomplishment of the discretization is to capture the intricate dependence of the energy spectrum on the worldsheet momenta p_k, which is seen, for example, in eq. (5.15). This information is washed out in the thermodynamic limit. Furthermore, we point out that one does not need to "discretize" the relativistic formulas for the rapidities $\varphi(p_k)$ or dispersion relations $q_r(p_k)$ from eqs. (6.8) and (6.10) to obtain the correct energy spectrum in the near-pp-wave limit.[13] It has been suggested [47, 48], however, that the proper "lattice" relations are indeed obtained by replacing the quantities $\varphi(p_k)$ and $q_r(p_k)$ with the following expressions, motivated by studies in the dual gauge theory:

$$\begin{aligned} \varphi(p_k) &= \frac{1}{2}\cot\left(\frac{p_k}{2}\right)\sqrt{1 + 8g^2\sin^2(p_k/2)}\,, \\ q_r(p_k) &= \frac{2\sin\left(\frac{r-1}{2}p_k\right)}{r-1}\left(\frac{\sqrt{1+8g^2\sin^2(p_k/2)} - 1}{2g^2\sin(p_k/2)}\right)^{r-1} . \end{aligned} \qquad (6.16)$$

Rather remarkably, Hofman and Maldacena have recently made contact with these expressions from the string side of the correspondence [52]. (It should be noted that [62] was an important precursor to this work; see also [53] for related developments).

Finally, we note that, with respect to strictly reproducing the energy shift in eq. (5.15), the two-body scattering phase in eq. (6.14) is not unique. Relying on near-pp-wave spectral information alone, it is straightforward to formulate an ad hoc scattering phase that succeeds in reproducing eq. (5.15):

$$\theta(p_k, p_j, \gamma) \approx -\frac{2}{\varphi(p_k) - \varphi(p_j)} - \left(2 + 4\pi\gamma\,\frac{p_k + p_j}{p_k\, p_j}\right)\sum_{r=1}^{\infty}\theta_r(p_k, p_j)\,. \qquad (6.17)$$

This expression, however, is not connected in any transparent way to the thermodynamic Bethe ansatz in eqs. (4.33), (4.34) above.

[12] Roughly speaking, this might be understood as arising from the fact that the TsT deformation considered here involves a shift in the timelike direction φ_1 [61].

[13] We thank Juan Maldacena for clarification on this point.

7. Comparison with the gauge theory

The construction of the Lunin-Maldacena solution has lead to a substantial amount of research on the dual β-deformed gauge theory (recent work includes [63–69]). The relationship of these theories to the spin-chain description was studied in [25, 26], and Bethe equations for the full twisted $\mathcal{N}=4$ SYM theory were constructed in [30]. For the deformed "$\mathfrak{su}(2)_\beta$" sector we need only consider a single type of impurity in the (twisted) Bethe equations:

$$1 = e^{-2\pi i\beta L}\left(\frac{u_k - i/2}{u_k + i/2}\right)^L \prod_{j\neq k}^{M} \frac{u_k - u_j + i}{u_k - u_j - i}\,, \tag{7.1}$$

where the cyclicity condition appears as

$$1 = e^{2\pi i\beta M}\prod_{j=1}^{M}\frac{u_k + i/2}{u_k - i/2}\,. \tag{7.2}$$

The corresponding equations to all loop-order in λ are given by applying the same twist to the Bethe ansatz formulated by Beisert, Dippel and Staudacher (BDS) [60] (these equations are still understood to be conjectural beyond three-loop order). It is then straightforward to compare $O(1/J)$ corrections to the string energy spectrum with corresponding predictions from the gauge theory spin chain. For the $\mathfrak{su}(2)_\beta$ sector we find (as is now expected) agreement up to three-loop order in λ' and disagreement at higher loops. As noted above, a Bethe ansatz for the twisted $\mathfrak{sl}(2)_\beta$ sector was also proposed in [30] by again taking a simple twist, analogous to the $\mathfrak{su}(2)_\beta$ sector. This is indeed what we find from the string theory at one-loop order in the $\mathfrak{sl}(2)_\gamma$ sector, but starting at two loops we find a more complicated dependence on the deformation parameter.

It was recently suggested [33] that the undeformed $\mathfrak{su}(2)$ sector of the gauge theory might be described non-perturbatively by the Hubbard model. (More precisely, the BDS Bethe equations arise in the weak-coupling limit of the Hubbard model, so the correctness of this approach is tied to that of the BDS equations.) Here we briefly describe how to modify this model to provide an analogous description in the β-deformed gauge theory.[14] Starting from the Hubbard model Hamiltonian

$$H_{\text{Hubbard}} = -t\sum_{n,\ \sigma=\uparrow,\downarrow}^{L}(c^\dagger_{n,\sigma}c_{n+1,\sigma} + c^\dagger_{n+1,\sigma}c_{n,\sigma}) + tU\sum_{n=1}^{L}c^\dagger_{n,\uparrow}c_{n,\uparrow}c^\dagger_{n,\downarrow}c_{n,\downarrow}\,, \tag{7.3}$$

we wish to study the following generalization:

$$H_{\text{Deformed}} = -t\sum_{n,\ \sigma=\uparrow,\downarrow}^{L}(f(n,\sigma)c^\dagger_{n,\sigma}c_{n+1,\sigma} + \tilde{f}(n,\sigma)c^\dagger_{n+1,\sigma}c_{n,\sigma}) + tU\sum_{n=1}^{L}c^\dagger_{n,\uparrow}c_{n,\uparrow}c^\dagger_{n,\downarrow}c_{n,\downarrow} \tag{7.4}$$

[14] Following the completion of this work, we were notified that similar results are derived in [70, 71].

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Now, exactly as in [33], one may calculate the effective action in this twisted model to first order in perturbation theory. Briefly, one takes the following Hamiltonian at leading order

$$H_0 = tU \sum_{n=1}^{L} c^\dagger_{n,\uparrow} c_{n,\uparrow} c^\dagger_{n,\downarrow} c_{n,\downarrow}\,, \tag{7.5}$$

and considers a subspace of the complete Fock space spanned by the states $c^\dagger_{1\sigma_1} c^\dagger_{2\sigma_2} \ldots c^\dagger_{L\sigma_L}|0\rangle$. It is then straightforward to show that the effective one-loop Hamiltonian in this particular subspace is

$$\begin{aligned} h = &-\frac{1}{2}\sum_n 2\left(f(n,\uparrow)\tilde{f}(n,\uparrow) + f(n,\downarrow)\tilde{f}(n,\downarrow)\right)\left((S^z_n S_{n+1})^z - \frac{1}{4}\right) \\ &+ \left(f(n,\uparrow)\tilde{f}(n,\uparrow) + f(n,\downarrow)\tilde{f}(n,\downarrow)\right)\left(S^z_{n+1} - S^z_n\right) \\ &+2f(n,\uparrow)\tilde{f}(n,\downarrow)S^+_n S^-_{n+1} + 2f(n,\downarrow)\tilde{f}(n,\uparrow)S^-_n S^+_{n+1}\,. \end{aligned} \tag{7.6}$$

We have used

$$S^+_n = c^\dagger_{n,\uparrow} c_{n,\downarrow}\,, \qquad S^-_n = c^\dagger_{n,\downarrow} c_{n,\uparrow}\,, \tag{7.7}$$

along with the following:

$$S^z_z = \frac{1}{2}\left(c^\dagger_{n,\uparrow}c_{n,\uparrow} - c^\dagger_{n,\downarrow}c_{n,\downarrow}\right) \simeq c^\dagger_{n,\uparrow}c_{n,\uparrow} - \frac{1}{2} \simeq \frac{1}{2} - c^\dagger_{n,\downarrow}c_{n,\downarrow}\,, \tag{7.8}$$

which only hold true when acting on singly-occupied states. One may then compare this Hamiltonian with the one-loop β-deformed $\mathfrak{su}(2)_\beta$ spin-chain Hamiltonian, formulated in [9]:

$$\begin{aligned} H = \frac{|h|^2}{2}\sum_{n=1}^{L}\Big(&\cosh 2\pi\kappa_d\left(S^z_n S^z_{n+1} - 1/4\right) + 1/2\sinh 2\pi\kappa_d\left(S^z_n - S^z_{n+1}\right) \\ &+\frac{1}{2}e^{2\pi i\beta}S^+_n S^-_{n+1} + \frac{1}{2}e^{-2\pi i\beta}S^-_n S^+_{n+1}\Big)\,. \end{aligned} \tag{7.9}$$

This form is slightly more general than we require for comparison with the γ-deformed string theory, for which we can set $|h| = 1$ and $\kappa_d = 0$. By matching coefficients

$$\begin{aligned} f(m,\uparrow) &= |h|e^{i\pi(\beta+\kappa_d)}\,, & \tilde{f}(m,\downarrow) &= |h|e^{i\pi(\beta-\kappa_d)}\,, \\ f(m,\downarrow) &= |h|e^{-i\pi(\beta+\kappa_d)}\,, & \tilde{f}(m,\uparrow) &= |h|e^{-i\pi(\beta-\kappa_d)}\,, \end{aligned} \tag{7.10}$$

we precisely reproduce the β-deformed $\mathfrak{su}(2)_\beta$ spin chain Hamiltonian in eq. (7.9).

We note that the β-deformed Hamiltonian with $|h| = 1$ can be reached from the undeformed model with twisted boundary conditions under

$$\begin{aligned} c_{m,\uparrow} &\to c_{m,\uparrow}e^{i\pi\beta m}e^{\pi\kappa_d/2}\,, \\ c_{m,\downarrow} &\to c_{m,\downarrow}e^{-i\pi\beta m}e^{-\pi\kappa_d/2}\,. \end{aligned} \tag{7.11}$$

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The matrix generating this transformation is

$$U = \begin{pmatrix} e^{i\pi\beta m} e^{\pi\kappa_d/2} & 0 \\ 0 & e^{-i\pi\beta m} e^{-\pi\kappa_d/2} \end{pmatrix}, \tag{7.12}$$

or, equivalently,

$$U = \exp\left(\frac{i\pi}{2}\sum_m (c^\dagger_{m,\uparrow}c_{m,\uparrow} - c^\dagger_{m,\downarrow}c_{m,\downarrow})(\beta m - i\kappa_d/2)\right). \tag{7.13}$$

In the case with $|h| = 1$, $\kappa_d = 0$, this transformation is unitary and the deformed Hubbard model merely corresponds to imposing twisted boundary conditions on the corresponding spin chain, with different conditions for the spin-up and spin-down fermions.

The Bethe equations solving this deformed Hubbard model arise as simple twists of the undeformed Lieb-Wu [72–74] equations. Twisted Lieb-Wu equations of this sort were studied by Yue and Deguchi in [75] (see also appendix C of [33], to which we also refer for notation):

$$\begin{aligned} e^{i\tilde{q}_n L} &= \prod_{j=1}^{M} \frac{u_j - \sqrt{2}g\sin(\tilde{q}_n + \phi_\uparrow) - i/2}{u_j - \sqrt{2}g\sin(\tilde{q}_n + \phi_\uparrow) + i/2}, \qquad n = 1, \ldots, N \\ \prod_{k=1}^{N} \frac{u_k - \sqrt{2}g\sin(\tilde{q}_n + \phi_\uparrow + i/2}{u_k - \sqrt{2}g\sin(\tilde{q}_n + \phi_\uparrow) - i/2} &= e^{iL(\phi_\downarrow - \phi_\uparrow)} \prod_{j\neq k}^{M} \frac{u_k - u_j + i}{u_k - u_j - i}. \end{aligned} \tag{7.14}$$

The energy eigenvalues are computed by solving the Bethe equations for the momenta $\tilde{q}_n$ and using the formula

$$E = \frac{\sqrt{2}}{g}\sum_{n=1}^{L}\cos(\tilde{q}_n + \phi_\uparrow). \tag{7.15}$$

As in [33], we have chosen Hubbard couplings that make the connection to the gauge theory obvious. For our twisted Hubbard model we set $\phi_\uparrow = \pi\beta$ and $\phi_\downarrow = -\pi\beta$, and it is straightforward to see that at half-filling (i.e., when the number of fermions N equals the lattice length L) and in the weak coupling limit ($g \to 0$), the second equation in (7.14) reduces to the one-loop twisted Bethe equations for the gauge theory in the $\mathfrak{su}(2)_\beta$ sector.

To see that the energy spectrum behaves as expected under this twist, it is useful to perform the transformation introduced in [33], similar to a Shiba transformation. With the definitions

$$\begin{aligned} c_{n,\circ} &= c^\dagger_{n,\uparrow}, \qquad & c^\dagger_{n,\circ} &= c_{n,\uparrow}, \\ c_{n,\updownarrow} &= c_{n,\downarrow}, \qquad & c^\dagger_{n,\updownarrow} &= c^\dagger_{n,\downarrow}, \end{aligned} \tag{7.16}$$

we rewrite the twisted Hamiltonian in its dual form

$$\begin{aligned} H = & \frac{1}{\sqrt{2}g} \sum_{n=1,\ \sigma=\circ,\updownarrow}^{L} \left(e^{i\phi_\sigma} c^\dagger_{n,\sigma}c_{n+1,\sigma} + e^{-i\phi_\sigma} c^\dagger_{n+1,\sigma}c_{n,\sigma}\right) \\ & - \frac{1}{g^2}\sum_{n=1}^{L}\left(1 - c^\dagger_{n,\circ}c_{n,\circ}\right) c^\dagger_{n,\updownarrow}c_{n,\updownarrow}, \end{aligned} \tag{7.17}$$

where $\phi_{\updownarrow} = \phi - \pi\beta,\ \phi_\circ = \pi - (\phi + \pi\beta)$. The parameter ϕ is analogous to Aharonov-Bohm flux [33]: it is chosen to be $\phi = 0$ for $L =$ odd and $\phi = \pi/2L$ for $L =$ even. The Bethe equations for the dual Hamiltonian at half filling take the form

$$e^{iL(\tilde{q}_n+\pi\beta)} = \prod_{j=1}^{M} \frac{u_j - \sqrt{2}g\sin(\tilde{q}_n - \phi) - i/2}{u_j - \sqrt{2}g\sin(\tilde{q}_n - \phi) + i/2}\,, \quad n = 1, \dots, 2M\,, \tag{7.18}$$

$$\prod_{n=1}^{2M} \frac{u_k - \sqrt{2}g\sin(\tilde{q}_n - \phi) + i/2}{u_k - \sqrt{2}g\sin(\tilde{q}_n - \phi) - i/2} = -\prod_{j\neq k}^{M} \frac{u_k - u_j + i}{u_k - u_j - i}\,, \tag{7.19}$$

where the energy is now given by

$$E = -\frac{M}{g^2} - \frac{\sqrt{2}}{g}\sum_{n=1}^{2M}\cos(\tilde{q}_n - \phi)\,. \tag{7.20}$$

It is straightforward to study the effect of the β deformation on this Hubbard model: one route is to solve the one-magnon problem, as in [33], with a system composed of $M = 1$ down spins and $L - 1$ up spins. Following [33], we adopt the ansatz

$$\tilde{q}_1 - \phi = \frac{\pi}{2} + q + i\delta\,, \quad \tilde{q}_2 - \phi = \frac{\pi}{2} + q - i\delta\,, \tag{7.21}$$

where δ parameterizes the binding of the quasi-momenta $\tilde{q}_n$. We may then use the set of one-magnon Bethe equations

$$e^{iL(\tilde{q}_1+\pi\beta)} = \frac{u - \sqrt{2}g\sin(\tilde{q}_1 - \phi) - i/2}{u - \sqrt{2}g\sin(\tilde{q}_1 - \phi) + i/2}\,, \quad e^{iL(\tilde{q}_2+\pi\beta)} = \frac{u - \sqrt{2}g\sin(\tilde{q}_2 - \phi) - i/2}{u - \sqrt{2}g\sin(\tilde{q}_2 - \phi) + i/2}\,, \tag{7.22}$$

and

$$\frac{u - \sqrt{2}g\sin(\tilde{q}_1 - \phi) + i/2}{u - \sqrt{2}g\sin(\tilde{q}_1 - \phi) - i/2}\,\frac{u - \sqrt{2}g\sin(\tilde{q}_2 - \phi) + i/2}{u - \sqrt{2}g\sin(\tilde{q}_2 - \phi) - i/2} = -1\,, \tag{7.23}$$

to find

$$q = \frac{\pi}{L}(n - \beta L)\,, \qquad n = 0, 1, \dots, L - 1\,. \tag{7.24}$$

The magnon momentum is defined to be $p \equiv 2q$, so we see that the twist indeed amounts to a shift in the mode number n by βL. We can now show that the energy E, written as a dispersion law in p, yields the form we expect in the presence of the twist (this turns out to be a nontrivial issue). Rewriting the equations in eq. (7.22) as

$$\sqrt{2}g\ \sin(\tilde{q}_{1,2} - \phi) - u = \frac{1}{2}\cot\left(\frac{(\tilde{q}_{1,2} + \pi\beta)L}{2}\right)\,, \tag{7.25}$$

one finds that, expressed in terms of mode numbers, all instances of β drop out of this equation. By splitting into real and imaginary pieces, we therefore obtain

$$\begin{aligned} \sinh(\delta) &= \frac{\tanh(\delta L)}{2\sqrt{2}g\sin(q)}\,, \\ u &= \sqrt{2}g\cos(q)\cosh(\delta) + \frac{(-1)^n(-1)^{\frac{L+1}{2}}}{2\cosh(\delta L)}\,. \end{aligned} \tag{7.26}$$

In the limit $L \to \infty$, terms of the form $e^{-\delta L}$ are dropped, and, following [33], we find that the energy formula for the one-magnon system is

$$E = -\frac{1}{g^2} - \frac{2\sqrt{2}}{g}\sin(p/2)\cosh(\delta) = -\frac{1}{g^2} + \frac{1}{g^2}\sqrt{1 + 8g^2 \sin^2\left(\frac{\pi}{L}(n - \beta L)\right)}\,. \qquad (7.27)$$

The full M-magnon problem can be solved in a completely analogous fashion, in which case the quasi-momenta $\tilde{q}_n$ are split into two sets:

$$\begin{aligned} \tilde{q}_n - \phi &= s_n \frac{\pi}{2} + \frac{p_n}{2} + i\delta_n\,, \\ \tilde{q}_{n+M} - \phi &= s_n \frac{\pi}{2} + \frac{p_n}{2} - i\delta_n\,, \end{aligned} \qquad (7.28)$$

where $s_n = \mathrm{sign}(p_n)$, and $n = 1, \ldots, M$. Once again, we take $L \to \infty$ and obtain

$$\begin{aligned} e^{i\tilde{q}_n L} &\sim e^{-\delta_n L} \to 0\,, \\ e^{i\tilde{q}_{n+M} L} &\sim e^{\delta_{n+M} L} \to \infty\,. \end{aligned} \qquad (7.29)$$

Thus, for L large, there exists a u denoted by u_n for each $n \in 1, \ldots, M$, such that

$$u_n - i/2 = \sqrt{2} g \sin(\tilde{q}_n - \phi)\,, \qquad u_n + i/2 = \sqrt{2} g \sin(\tilde{q}_{n+M} - \phi)\,, \qquad (7.30)$$

exactly as in [33]. We can use these equations to to determine δ_n and u_n in terms of p_n, and we find the same expressions as in the undeformed case:

$$\sinh \delta_n = \frac{1}{2\sqrt{2} g |\sin \frac{p_n}{2}|}\,, \qquad (7.31)$$

$$u_n = \frac{1}{2}\cot\frac{p_n}{2}\sqrt{1 + 8g^2 \sin^2\frac{p_n}{2}}\,. \qquad (7.32)$$

To determine $\tilde{q}_n$ in terms of u_n, we multiply the n^{th} and $(n+M)^{\mathrm{th}}$ equation in (7.19) and it is straightforward to see that

$$e^{iL(p_n + 2\pi\beta)} = \prod_{j=1,\ j\neq n}^{M} \frac{u_n - u_j + i}{u_n - u_j - i}\,. \qquad (7.33)$$

We are therefore lead to conclude that the twisted BDS Bethe equations are properly encoded in this β-deformed Hubbard model, which, given that the deformation is merely a twisted boundary condition, is to be expected.

8. Conclusions

TsT transformations yield a simple deformation of the usual correspondence between string theory in $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM theory. Many of the recent developments stemming from the discovery of integrable structures in this correspondence are thus easily tested in this interesting new setting. In recent years, for example, a heuristic methodology has emerged for formulating quantum string Bethe equations from Lax representations of string sigma models. Thermodynamic Bethe equations emerge directly from the sigma model by formulating the Bethe ansatz as a Riemann-Hilbert problem. However, one must rely on detailed studies of the dual gauge theory for instruction on how to discretize these equations. In this paper we have studied whether the discretization procedure handed down from the gauge theory can be applied in the case of TsT-deformed string theory. We have shown that these rules can indeed be adopted under relatively dramatic deformations of the original problem, and we have been able to successfully reproduce $O(1/J)$ corrections to the plane-wave energy spectrum in deformed $\mathfrak{sl}(2)_\gamma$ and $\mathfrak{su}(2)_\gamma$ subsectors.

We have also made contact in the $\mathfrak{su}(2)_\gamma$ sector with a recent formulation of the all-loop gauge theory problem written as a low-energy effective theory embedded in the Hubbard model. One open problem is that it is difficult to make any such contact with the gauge theory side of the correspondence in the $\mathfrak{sl}(2)_\gamma$ sector. General considerations lead us to believe that the field theory dual to this deformed string theory is a non-commutative gauge theory. It would be very interesting to study whether a non-commutative deformation of $\mathcal{N} = 4$ SYM theory encodes some portion of the $\mathfrak{sl}(2)_\gamma$ string spectrum computed here.

Furthermore, one can extend the analysis of the near-pp-wave theory to larger subsectors of excitations. At one-loop order in λ' one might expect the full set of $\mathfrak{so}(6)$ bosons to comprise a closed subsector, sensitive to all three deformation parameters of the non-supersymmetric γ-deformed background. Even at one-loop order, however, string theory predictions disagree with corresponding anomalous dimensions in the gauge theory. It is possible that in the non-supersymmetric deformation the $\mathfrak{so}(6)$ bosons do not form a closed subsector but mix with the fermions. Another possibility is that the background itself needs to be corrected, perhaps along the lines of [76]. It is also interesting that the non-supersymmetric deformation is unstable due to the flow of double trace couplings [77], similar to the case found for orbifolds of $\mathcal{N} = 4$ SYM [78, 79]. For the deformed theory, however, the endpoint of the flow is unclear.

At this point there exist a number of quantum string Bethe equations harboring a great deal of predictive information that remains untested. In this regard it would obviously be valuable to obtain spectral information directly from the string theory at higher orders in the $1/J$ expansion. Attempts to study this difficult problem seem to be hindered by the lack of a suitable renormalization scheme for the Green-Schwarz formulation of the worldsheet lightcone field theory. Perhaps a covariant approach is needed to glean reliable information beyond the near-pp-wave limit. Alternatively, it would be extremely valuable to uniquely derive the complete S matrix of the worldsheet theory based on the underlying symmetries in the problem.

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References

[1] O. Lunin and J.M. Maldacena, *Deforming field theories with* $\mathrm{U}(1) \times \mathrm{U}(1)$ *global symmetry and their gravity duals*, *JHEP* **05** (2005) 033 [hep-th/0502086].

[2] R.G. Leigh and M.J. Strassler, *Exactly marginal operators and duality in four-dimensional* $N = 1$ *supersymmetric gauge theory*, *Nucl. Phys.* **B 447** (1995) 95 [hep-th/9503121].

[3] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[4] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109].

[5] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[6] R. de Mello Koch, J. Murugan, J. Smolic and M. Smolic, *Deformed pp-waves from the Lunin-Maldacena background*, *JHEP* **08** (2005) 072 [hep-th/0505227].

[7] T. Mateos, *Marginal deformation of* $N = 4$ *sym and Penrose limits with continuum spectrum*, *JHEP* **08** (2005) 026 [hep-th/0505243].

[8] M. Spradlin, T. Takayanagi and A. Volovich, *String theory in beta deformed spacetimes*, *JHEP* **11** (2005) 039 [hep-th/0509036].

[9] S.A. Frolov, R. Roiban and A.A. Tseytlin, *Gauge-string duality for superconformal deformations of* $N = 4$ *super Yang-Mills theory*, *JHEP* **07** (2005) 045 [hep-th/0503192].

[10] N.P. Bobev, H. Dimov and R.C. Rashkov, *Semiclassical strings in Lunin-Maldacena background*, hep-th/0506063.

[11] J. Plefka, *Spinning strings and integrable spin chains in the AdS/CFT correspondence*, hep-th/0507136.

[12] S. Ryang, *Rotating strings with two unequal spins in Lunin-Maldacena background*, *JHEP* **11** (2005) 006 [hep-th/0509195].

[13] R.C. Rashkov, K.S. Viswanathan and Y. Yang, *Generalization of the lunin-maldacena transformation on the* $AdS_5 \times S^5$ *background*, *Phys. Rev.* **D 72** (2005) 106008 [/].

[14] N.P. Bobev, H. Dimov and R.C. Rashkov, *Semiclassical strings, dipole deformations of* $N = 1$ *SYM and decoupling of kk modes*, *JHEP* **02** (2006) 064 [hep-th/0511216].

[15] S. Frolov, *Lax pair for strings in Lunin-Maldacena background*, *JHEP* **05** (2005) 069 [hep-th/0503201].

[16] L.F. Alday, G. Arutyunov and S. Frolov, *Green-Schwarz strings in TST-transformed backgrounds*, *JHEP* **06** (2006) 018 [hep-th/0512253].

[17] G. Mandal, N.V. Suryanarayana and S.R. Wadia, *Aspects of semiclassical strings in* AdS_5, *Phys. Lett.* **B 543** (2002) 81 [hep-th/0206103].

[18] I. Bena, J. Polchinski and R. Roiban, *Hidden symmetries of the* $AdS_5 \times S^5$ *superstring*, *Phys. Rev.* **D 69** (2004) 046002 [hep-th/0305116].

[19] V.A. Kazakov, A. Marshakov, J.A. Minahan and K. Zarembo, *Classical/quantum integrability in AdS/CFT*, *JHEP* **05** (2004) 024 [hep-th/0402207].

[20] V.A. Kazakov and K. Zarembo, *Classical/quantum integrability in non-compact sector of AdS/CFT*, *JHEP* **10** (2004) 060 [hep-th/0410105].

[21] N. Beisert, V.A. Kazakov, K. Sakai and K. Zarembo, *The algebraic curve of classical superstrings on $AdS_5 \times S^5$*, *Commun. Math. Phys.* **263** (2006) 659 [hep-th/0502226].

[22] N. Beisert, V.A. Kazakov and K. Sakai, *Algebraic curve for the SO(6) sector of AdS/CFT*, *Commun. Math. Phys.* **263** (2006) 611 [hep-th/0410253].

[23] S. Schafer-Nameki, *The algebraic curve of 1-loop planar $N = 4$ SYM*, *Nucl. Phys.* **B 714** (2005) 3 [hep-th/0412254].

[24] L.F. Alday, G. Arutyunov and A.A. Tseytlin, *On integrability of classical superstrings in $AdS_5 \times S^5$*, *JHEP* **07** (2005) 002 [hep-th/0502240].

[25] R. Roiban, *On spin chains and field theories*, *JHEP* **09** (2004) 023 [hep-th/0312218].

[26] D. Berenstein and S.A. Cherkis, *Deformations of $N = 4$ SYM and integrable spin chain models*, *Nucl. Phys.* **B 702** (2004) 49 [hep-th/0405215].

[27] L. Freyhult, C. Kristjansen and T. Mansson, *Integrable spin chains with* $\mathrm{U}(1)^3$ *symmetry and generalized Lunin-Maldacena backgrounds*, *JHEP* **12** (2005) 008 [hep-th/0510221].

[28] H.-Y. Chen and S. Prem Kumar, *Precision test of AdS/CFT in Lunin-Maldacena background*, *JHEP* **03** (2006) 051 [hep-th/0511164].

[29] H.-Y. Chen and K. Okamura, *The anatomy of gauge/string duality in Lunin-Maldacena background*, *JHEP* **02** (2006) 054 [hep-th/0601109].

[30] N. Beisert and R. Roiban, *Beauty and the twist: the Bethe ansatz for twisted $N = 4$ SYM*, *JHEP* **08** (2005) 039 [hep-th/0505187].

[31] A. Hashimoto and N. Itzhaki, *Non-commutative Yang-Mills and the AdS/CFT correspondence*, *Phys. Lett.* **B 465** (1999) 142 [hep-th/9907166].

[32] J.M. Maldacena and J.G. Russo, *Large-N limit of non-commutative gauge theories*, *JHEP* **09** (1999) 025 [hep-th/9908134].

[33] A. Rej, D. Serban and M. Staudacher, *Planar $N = 4$ gauge theory and the Hubbard model*, *JHEP* **03** (2006) 018 [hep-th/0512077].

[34] S.A. Frolov, R. Roiban and A.A. Tseytlin, *Gauge-string duality for (non)supersymmetric deformations of $N = 4$ super Yang-Mills theory*, *Nucl. Phys.* **B 731** (2005) 1 [hep-th/0507021].

[35] D. Berenstein, J.M. Maldacena and H. Nastase, *Strings in flat space and pp waves from $N = 4$ super Yang-Mills*, *JHEP* **04** (2002) 013 [hep-th/0202021].

[36] C.G. Callan Jr. et al., *Quantizing string theory in $AdS_5 \times S^5$: beyond the pp-wave*, *Nucl. Phys.* **B 673** (2003) 3 [hep-th/0307032].

[37] C.G. Callan Jr., T. McLoughlin and I.J. Swanson, *Holography beyond the Penrose limit*, *Nucl. Phys.* **B 694** (2004) 115 [hep-th/0404007].

[38] G. Arutyunov and S. Frolov, *Uniform light-cone gauge for strings in $AdS_5 \times S^5$: solving $su(1|1)$ sector*, *JHEP* **01** (2006) 055 [hep-th/0510208].

[39] J. Madore, S. Schraml, P. Schupp and J. Wess, *Gauge theory on noncommutative spaces*, *Eur. Phys. J.* **C 16** (2000) 161 [hep-th/0001203].

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[40] P.M. Ho and Y.T. Yeh, *Noncommutative D-brane in non-constant NS-NS B field background*, *Phys. Rev. Lett.* **85** (2000) 5523 [hep-th/0005159].

[41] P.M. Ho and S.P. Miao, *Noncommutative differential calculus for D-brane in non-constant B field background*, *Phys. Rev.* **D 64** (2001) 126002 [hep-th/0105191].

[42] L. Cornalba and R. Schiappa, *Nonassociative star product deformations for D-brane worldvolumes in curved backgrounds*, *Commun. Math. Phys.* **225** (2002) 33 [hep-th/0101219].

[43] G. Arutyunov and S. Frolov, *Integrable hamiltonian for classical strings on $AdS_5 \times S^5$*, *JHEP* **02** (2005) 059 [hep-th/0411089].

[44] T. McLoughlin and I.J. Swanson, *N-impurity superstring spectra near the pp-wave limit*, *Nucl. Phys.* **B 702** (2004) 86 [hep-th/0407240].

[45] I.J. Swanson, *Superstring holography and integrability in $AdS_5 \times S^5$*, hep-th/0505028.

[46] C.G. Callan Jr., T. McLoughlin and I.J. Swanson, *Higher impurity AdS/CFT correspondence in the near-BMN limit*, *Nucl. Phys.* **B 700** (2004) 271 [hep-th/0405153].

[47] G. Arutyunov, S. Frolov and M. Staudacher, *Bethe ansatz for quantum strings*, *JHEP* **10** (2004) 016 [hep-th/0406256].

[48] M. Staudacher, *The factorized S-matrix of CFT/ads*, *JHEP* **05** (2005) 054 [hep-th/0412188].

[49] N. Beisert and M. Staudacher, *Long-range $PSU(2,2|4)$ Bethe ansaetze for gauge theory and strings*, *Nucl. Phys.* **B 727** (2005) 1 [hep-th/0504190].

[50] N. Beisert, *The $su(2|2)$ dynamic S-matrix*, hep-th/0511082.

[51] R.A. Janik, *The $AdS_5 \times S^5$ superstring worldsheet S-matrix and crossing symmetry*, *Phys. Rev.* **D 73** (2006) 086006 [hep-th/0603038].

[52] D.M. Hofman and J.M. Maldacena, *Giant magnons*, hep-th/0604135.

[53] N. Dorey, *Magnon bound states and the AdS/CFT correspondence*, hep-th/0604175.

[54] G. Arutyunov, *Quantum strings and Bethe equations*, *Fortschr. Phys.* **53** (2005) 846 [hep-th/0412072].

[55] S. Schafer-Nameki, M. Zamaklar and K. Zarembo, *Quantum corrections to spinning strings in $AdS_5 \times S^5$ and Bethe ansatz: a comparative study*, *JHEP* **09** (2005) 051 [hep-th/0507189].

[56] N. Beisert and A.A. Tseytlin, *On quantum corrections to spinning strings and Bethe equations*, *Phys. Lett.* **B 629** (2005) 102 [hep-th/0509084].

[57] S. Schafer-Nameki and M. Zamaklar, *Stringy sums and corrections to the quantum string Bethe ansatz*, *JHEP* **10** (2005) 044 [hep-th/0509096].

[58] R. Hernandez and E. Lopez, *Quantum corrections to the string Bethe ansatz*, *JHEP* **07** (2006) 004 [hep-th/0603204].

[59] L. Freyhult and C. Kristjansen, *A universality test of the quantum string Bethe ansatz*, *Phys. Lett.* **B 638** (2006) 258 [hep-th/0604069].

[60] N. Beisert, V. Dippel and M. Staudacher, *A novel long range spin chain and planar $N = 4$ super Yang-Mills*, *JHEP* **07** (2004) 075 [hep-th/0405001].

[61] N. Beisert, 2006, private communication.

[62] D. Berenstein, D.H. Correa and S.E. Vazquez, *All loop BMN state energies from matrices*, *JHEP* **02** (2006) 048 [hep-th/0509015].

[63] U. Gursoy and C. Núñez, *Dipole deformations of $N = 1$ SYM and supergravity backgrounds with* $\mathrm{U}(1) \times \mathrm{U}(1)$ *global symmetry*, *Nucl. Phys.* **B 725** (2005) 45 [hep-th/0505100].

[64] D.Z. Freedman and U. Gursoy, *Comments on the beta-deformed $N = 4$ SYM theory*, *JHEP* **11** (2005) 042 [hep-th/0506128].

[65] S. Penati, A. Santambrogio and D. Zanon, *Two-point correlators in the beta-deformed $N = 4$ SYM at the next-to-leading order*, *JHEP* **10** (2005) 023 [hep-th/0506150].

[66] G.C. Rossi, E. Sokatchev and Y.S. Stanev, *New results in the deformed $N = 4$ SYM theory*, *Nucl. Phys.* **B 729** (2005) 581 [hep-th/0507113].

[67] A. Mauri, S. Penati, A. Santambrogio and D. Zanon, *Exact results in planar $N = 1$ superconformal Yang-Mills theory*, *JHEP* **11** (2005) 024 [hep-th/0507282].

[68] V.V. Khoze, *Amplitudes in the beta-deformed conformal Yang-Mills*, *JHEP* **02** (2006) 040 [hep-th/0512194].

[69] G. Georgiou and V.V. Khoze, *Instanton calculations in the beta-deformed AdS/CFT correspondence*, *JHEP* **04** (2006) 049 [hep-th/0602141].

[70] P.B. Ramos and M.J. Martins, *Algebraic bethe ansatz approach for the one-dimensional Hubbard model*, *J. Phys.* **A30** (1997) L195–L202 [hep-th/9605141].

[71] M.J. Martins and P.B. Ramos, *The quantum inverse scattering method for hubbard-like models*, *Nucl. Phys.* **B 522** (1998) 413 [solv-int/9712014].

[72] E.H. Lieb and F.Y. Wu, *Absence of mott transition in an exact solution of the short-range, one-band model in one dimension*, *Phys. Rev. Lett.* **20** (1968) 1445.

[73] E.H. Lieb and F.Y. Wu, *Absence of mott transition in an exact solution of the short-range, one-band model in one dimension*, *Phys. Rev. Lett.* **21** (1968) 192.

[74] E.H. Lieb and F.Y. Wu, *The one-dimensional Hubbard model: a reminiscence*, *Physica A* **321** (2003) 1.

[75] R. Yue and T. Deguchi, *Analytic Bethe ansatz for 1D Hubbard model and twisted coupled XY model*, cond-mat/9606039.

[76] C.-S. Chu and V.V. Khoze, *String theory dual of the beta-deformed gauge theory*, *JHEP* **07** (2006) 011 [hep-th/0603207].

[77] A. Dymarsky and R. Roiban, 2006, private communication.

[78] A. Dymarsky, I.R. Klebanov and R. Roiban, *Perturbative search for fixed lines in large-N gauge theories*, *JHEP* **08** (2005) 011 [hep-th/0505099].

[79] A. Dymarsky, I.R. Klebanov and R. Roiban, *Perturbative gauge theory and closed string tachyons*, *JHEP* **11** (2005) 038 [hep-th/0509132].

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